Jnanabha Vol. 30 2000 No. 1

ISSN 0304-9892

Jñānābha ज्ञानाभ

Exchange Copy

Library (99)

6/03

S L Singh

19.6.03

VOLUME 30

2000

Published by :

The Vijñāna Parishad of India

DAYANAND VEDIC POSTGRADUATE COLLEGE

(Bundelkhand University)

ORAI, U.P., INDIA

ISSN 0304-9892

Jñānābha ज्ञानाभ

VOLUME 30

2000

Published by :

The Vijñāna Parishad of India

DAYANAND VEDIC POSTGRADUATE COLLEGE

(Bundelkhand University)

ORAI, U.P., INDIA

ISSN 0304-9892

Jñānābha ज्ञानाभ

VOLUME 30

2000

Published by :

The Vijñāna Parishad of India

DAYANAND VEDIC POSTGRADUATE COLLEGE

(Bundelkhand University)

ORAI, U.P., INDIA

CONTENTS

CONTAENTS

CONTAENTS

INFORMATION FOR CONTRIBUTORS/SUBSCRIBERS

Jñānābha is published annualy. It is an interdisciplinary journal devoted primarily to research articles in *all* areas of the mathematical, physical and biological sciences; it does, however, encourage original mathematical works which are motivated by and relevant to applications in the social, management, biomedical or engineering sciences. Papers intended for publication in this journal should be in typed form or offset-reproduced (*not dittoed*), double spaced with generous margins and they may be written in Hindi or English. Manuscripts in duplicate, if possible, may be submitted to *either* of the Editors. *It is mandatory for every author in* Jñānābha *to be a member of the Vijñāna Parishad of India in good standing and to pay publication charges which will be based upon the number of printed pages.*

The submission of a paper implies the author's assurance that the paper has not been widely circulated, copyrighted, published or submitted for publication elsewhere.

Authors are advised to submit *only* neatly (and carefully) type-written and thoroughly checked manuscripts employing accepted conventions of references, notations, displays, etc. ; *all* typescripts not in a format suitable for publication in this journal will be returned *unrefereed.*

25 reprints are provided free for each article ; additional reprints, in multiples of 25 may be obtained at cost price, if the authors give prior intimation while returning proofs.

Effective with Volume 23 the price per volume is Rs. 100.00 (or U.S. $25.00). Individual members of the Vijnāna Parishad of India are entitled to free subscriptions to the current issues of this journal. Individual membership : Rs. 50.00 (or U.S. $20.00) per calender year; Life membership : Rs. 500.00 (or U.S. $ 200.00) Back volumes are available at special price. (*Payment of dues by cheques must include appropriate bank charges.)*

[By a reciprocity agreement with the **American Mathematical Society**, an individual/life member of the Parishad residing outside North American continent may join the Society by submitting an application for membership on the form that may be obtained from the office of the Society **(P.O. Box 6248, Providence, Rhode, Island 02940, U.S.A.)** and by paying the Society's current dues as a *reciprocity member* at a considerably reduced rate; the usual requirements that the applicant be endorsed by two members of the Society and that the candidate be elected by the Council are waived, but this reduction in dues to the Society does not apply to those individual/life members of the Parishad who reside, even on a temporary basis, in the North American area (i.e., U.S.A. and Canada).

The mathematical content of this journal is indexed in the Society's biweekly **Current Mathematical Publications** and is reviewed, among others, by **Mathematical Reviews** (U.S.A), **Zentralblatt für Mathematik** (Germany) and **Applied Mechanics Reviews** (U.S.A.).]

All communications regarding subscriptions, order for back volumes, membership of the Vijñāna Parishad of India, change of address, etc. and all books for review, should be addressed to :

The Secretary,
Vijñāna Parisahd of India
D.V. Postgraduate College,
Orai-285001, U.P., India
Ph. : 05162-251607

Jñānābha, Vol. 30, 2000

EXISTENCE OF MAXIMAL ELEMENTS AND APPLICATIONS

By

A. Carbone

Dipartimento di Matematica

Università della Calabria- 87036 Arcavacata di Rende (Cosenza), Italy

Email : carbonea@unical.it

and

S.P. Singh

Department of Mathematics and Statistics

Memorial University of Newfoundland St. John's, NF, Canada, AIC 5S7

Email : spsingh@math.mun.ca

(Received : December 31, 2000)

Recall that a binary relation F on a set C is a subset of $C \times C$ or a mapping of C into itself. It is written yFx or $y \in Fx$ to mean that y stands in relation F to x.

A maximal element of F is a point x such that no point y satisfies $y \in Fx$, i.e., $Fx = \phi$. Thus the set of maximal elements is

$$\{x \in C : Fx = \phi\} = \bigcap_{x \in C} (C \setminus F^{-1}x),$$

where

$$F^{-1}x = \{y \in C : x \in Fy\}.$$

The following well known theorem is due to Sonnenschein [13].

Theorem 1 *Let C be a compact convex subset of R^n and F a binary relation on C satisfying*

(i) $x \notin coFx$ *for all* $x \in C$ *(co stands for convex hull),*

(ii) *if* $y \in F^{-1}x$, *then there exists some* $x_1 \in C$ *(possibly* $x_1 = x$*) such that* $y \in intF^{-1}x_1$.

Then F has a maximal element.

Bergstrom proved the following [1].

Theorem 2 *Let C be a nonempty, compact convex subset of R^n and $F: C \to 2^C$ a preference map satisfying*

(i) $x \notin coFx$ *for each* $x \in C$,

(ii) F *is lower semicontinuous on* C.

Then F has a maximal element.

We note that the results on maximal elements are useful in fixed

point theory, variational inequalities, complementarity problems and best approximation (see [2] and [12]).

Ky Fan's Lemma [6]. *Let C be a nonempty compact convex subset of R^n and $F : C \to 2^C$ a multifunction such that*

(i) Fx is convex for each $x \in C$,

(ii) F has open graph,

(iii) $x \notin Fx$ for each $x \in C$.

Then F has a maximal element.

We prove the following for maximal elements :

Theorem 3. *Let C be a nonempty closed convex subset of a Hausdorff topological vector space X and $F:C \to 2^C$ satisfy*

(i) $x \notin Fx$ for each $x \in C$,

(ii) Fx is closed for each $x \in C$,

(iii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ is convex for each $y \in C$,

(iv) C can be covered by some finite number of closed sets Fx_1, Fx_2,..., Fx_n.

Then F has a maximal element.

Proof.

Let $Fx \neq \phi$ for each $x \in C$. Define $G(x) = C \setminus Fx$. Then $G(x)$ is open for each $x \in C$ since Fx is closed. By (iv) $C = \cup_{i=1}^{n} Fx_i$, so $\cap_{i=1}^{n} Gx_i = \cap_{i=1}^{n} (C/Fx_i) = (\cup_{i=1}^{n} Fx_i)^c = \phi$. Therefore G is not a KKM-map.

Hence, there exists a finite set $\{x_1, x_2, ..., x_k\}$ of C such that

$$z = \sum_{i=0}^{k} \lambda_i x_i \not\subset \overset{k}{\underset{1}{\cup}} Gx_i,$$

where $\lambda_i = 0$ and $\sum \lambda_i = 1$.

so $z \in \cap_{i=1}^{k} Fx_i$ and $x_i \in F^{-1}z$, $i = 1, 2, ..., k$. Since $F^{-1}z$ is convex so

$$z = \sum_{i=0}^{k} \lambda_i x_i \in F^{-1}z$$

implying that $z \in Fz$, a contradiction to hypothesis (i). So $Fx = \phi$.

The following result is a corollary :

Corollary 1. *Let C be a nonempty closed convex subset of a topological vector space X and $F:C \to 2^C$ an upper semicontinuous convex-valued map. Further, assume that there is some finite subset B of C such that $Fx \cap B \neq \phi$ for every $x \in C$, and $x \notin Fx$ for each $x \in C$. Then F has a maximal element.*

Proof.

Let $G : C \to 2^C$ be defined by

$$Gx = F^{-1}x = \{y \in C : x \in Fy\}.$$

Since F is upper semicontinuous, each $G(x)$ is closed. Now, $G^{-1}(y) = (F^{-1}y)^{-1} = Fy$ is convex.

Also C can be covered by $\{G(x) : x \in B\}$, finitely many closed sets.

By hypothesis $x \notin Fx$ for $x \in C$. So by Theorem 3, F has a maximal element.

In the following, the KKM-map principle is applied :

Theorem 4. *Let C be a nonempty convex subset of a topological vector space X, and $F: C \to 2^C$ satisfy*

(i) $x \notin coFx$ *for each* $x \in C$,

(ii) if $y \in F^{-1}x$, *then there exists some* $x_1 \in C$ *(possibly* $x_1 = x$*) such that* $y \in intF^{-1}x_1$,

(iii) C has a nonempty compact convex subset D such that the set

$$B = \{x \in C : y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

Proof. Define $G(x) = C \setminus int\, F^{-1}(x)$ for each $x \in C$. Then $G(x)$ is closed for each $x \in C$. We claim that G is a KKM-map. Let $z \in co(x_1, x_2, ..., x_n)$. *If* $z \notin \cup_{i=1}^{n} Gx_i$ then $z \notin Gx_i$, $i = 1,2,...n$, that is, $z \in F^{-1}x_i$, $i = 1,2,...,n$. Thus $x_i \in Fz$ and $z \in coFz$, contradiction to (i). Hence G is a KKM-map.

Now,

$$\begin{aligned} B &= \{x \in C : y \notin Fx \text{ for all } y \in D\} \\ &= \{x \in C : x \notin F^{-1}y \text{ for all } y \in D\} \\ &= \{x \in C : x \in Gy \text{ for all } y \in D\} \\ &= \cap_{y \in D} Gy \neq \phi, \end{aligned}$$

that is, $x_0 \in \cap_{y \in D} Gy$.

Thus $x_0 \notin F^{-1}y$ for all $y \in C$, that is, $y \notin Fx_0$ for all $y \in C$ and $Fx_0 = \phi$.

The following results are derived as corollaries :

Corollary 2. *Let C be a nonempty convex subset of a topological vector space E and $F : C \to 2^C$ satisfy:*

(i) $x \notin Fx$,

(ii) Fx *is convex for each* $x \in C$,

(iii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ *is open in C for each* $y \in C$,

(iv) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C: y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

The following is due to Lin [9,10]:

Corollary 3. *Let C be a nonempty subset of a topological vector space E and $F : C \to 2^C$ satisfy*

(i) $x \notin co\, F(x)$ *is convex for each* $x \in C$,

(ii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ *is open for each* $y \in C$,

(iii) *C has a nonempty compact convex subset D such that the set*

$$B = \{x \in C: y \notin Fx \text{ for all } y \in D\}$$

is compact.

Then F has a maximal element.

In case C is a nonempty compact convex subset of a topological vector space X and $F : C \to 2^C$ satisfy

(i) $x \notin coF(x)$ for each $x \in C$,

(ii) $F^{-1}(y) = \{x \in C : y \in Fx\}$ is open for each $y \in C$,

then F has a maximal element.

Corollary 4. *Let C be a nonempty compact convex subset of a normed linear space X and $f : C \to X$ a continuous map. Then there is a $y_0 \in C$ such that*

$$||y_0 - fy_0|| = d(fy_0, C),$$

where $d(x,C) = \inf \{||x - y|| : y \in C\}$.

The following variant of Hartman and Stampacchia [7] theorem is very useful in variational problems.

Theorem 5. *Let C be a closed bounded convex subset of a Hilbert space H. If $f : C \to H$ is a continuous and monotone map, then there exists a $y_0 \in C$ such that $\langle fy_0, y_0 - x\rangle \leq 0$ for all $x \in C$.*

Recall that f is monotone on C if $\langle fx - fy, x - y \rangle \geq 0$ for all $x, y \in C$.

The fixed point theorem due to Browder [3] follows as a corollary.

Corollary 5. *If C is a closed bounded convex subset of a Hilbert space H and $f : C \to C$ a nonexpansive map then f has a fixed point.*

Put $F = 1 - f$. Then $F : C \to H$ is continuous and monotone; hence there is a $y_0 \in C$ such that $(Fy_0, y_0 - x) \leq 0$ for all $x \in C$, i.e., $< (1-f)\, y_0,$

$$||fx_0 - y|| = ||fx_0 - \lambda x_0 - (1-\lambda)fx_0||$$
$$= ||\lambda|| \, ||x_0 - fx_0|| < ||x_0 - fx_0||,$$

a contradiction, so $x_0 = fx_0$.

Using Fan's best approximation theorem, one could prove the following variational inequality due to Hartman and Stampacchia [7].

Theorem 10. Let *C be a compact convex subset of* R^n *and* $f : C \to R^n$ *a continuous function.*

Then there exists an $x_0 \in C$ *such that*

$$< fx_0 , y - x_0 > \geq 0 \text{ for all } y \in C.$$

Proof. Let $g = I - f: C \to R^n$. Then g is a continuous function. By Theorem 9, there exists a $y_0 \in C$ such that

$$||gy_0 - y_0|| \leq ||gy_0 - x|| \text{ for all } x \in C,$$

that is,

$$< gy_0 - y_0, y_0 - x> \geq 0 \text{ for all } x \in C.$$

Thus

$$< y_0 - fy_0 - y_0, y_0 - x> \geq 0 \text{ for all } x \in C,$$

that is

$$<-fy_0 , \; y_0 - x > \geq 0 \text{ for all } x \in C,$$

Hence

$$<fy_0 , \; x - y_0 > \geq 0 \text{ for all } x \in C.$$

Note. If $f = I - g$, then

$<y_0 - gy_0, \; x - y_0 > \geq 0$ for all $x \in C$,

gives that g has a fixed point by taking $x = gy_0$.

Theorem 9 is useful in proving the existence of zeros of a given function.

Theorem 11. *Let C be a compact convex subset of* R^n *and* $f : C \to R^n$ *a continuous function. Let* $x - \lambda fx \in C$ *for all* $x \in C$ *and for some* $\lambda > 0$.

Then there is an $x_0 \in C$ *such that* $fx_0 = 0$ *[14].*

Proof. Let $gx = x - \lambda fx$. Then by Theorem 9 there is an $x_0 \in C$ such that

$$||gx_0 - x_0|| = d(gx_0 , C).$$

In the case $x - \lambda \, fx \in C$ for all $x \in C$, then $gx_0 \in C$ and $gx_0 = x_0$, that is, $x_0 - \lambda \, fx_0 = x_0$, so $fx_0 = 0$.

REFERENCES

[1] T. Bergstrom, The existence of maximal elements and equilibria in the absence of transitivity, *University of Michigan*, 1975.

[2] K.C. Border. *Fixed Point Theorems with Applications to Economics and Game Theory*, Cambridge University Press, London, 1985.

$z \in Pz$. This contradicts assumption (i) and hence P has a maximal element.

The following result due to Ky Fan [5] is proved by using the existence of maximal elements.

Theorem 8. *Let C a compact convex subset of R^n and let $A \subset C \times C$ be a closed set such that*

(i) *for each $x \in C$, $(x,x) \in A$,*

(ii) *for all $y \in C$, $\{x \in C : (x,y) \notin A\}$ is convex or ϕ.*

Then there is a $\bar{y} \in C$ such that $C \times \{\bar{y}\} \subset A$.

Proof. Define F on C by

$$Fy = \{x \in C : (x,y) \notin A\}$$

Then $y \notin Fy$, and Fy is convex for each y.

F has open graph since $(x,y) \notin A$ so $(x,y) \in A^c$ is an open set. So by Ky Fan's Lemma F has a maximal element.

Hence, there exists a $\bar{y} \in C$ such that $F\bar{y} = \phi$, that is, $C \times \{\bar{y}\} \subset A$.

We prove the best approximation theorem by making use of maximal elements and then derive a few fixed point theorems.

The following is Fan's best approximation theorem [6]:

Theorem 9. *Let C be a compact convex subset of R^n and $f : C \to R^n$ a continuous function. Then there exists an $x_0 \in C$ such that*

$$||x_0 - fx_0|| = d(fx_0, C) \leq ||fx_0 - y|| \text{ for all } y \in C.$$

Proof. Define F on C by

$$y \in Fx \text{ if and only if } ||y - fx|| < ||x - fx||.$$

Therefore, by Ky Fan's Lemma we get that F has a maximal element. Since Fx is convex, $x \notin Fx$ and F has open graph; that is, $F\bar{x} = \phi$ for $\bar{x} \in C$. Hence

$$||\bar{x} - f\bar{x}|| \leq ||f\bar{x} - y|| \text{ for all } y \in C.$$

Note. In case $f : C \to C$ in Theorem 9, then f has a fixed point.

In Theorem 9, if $\bar{x} \neq f\bar{x}$, then the following condition will guarantee that f has a fixed point.

If $\bar{x} \neq f\bar{x}$, then the line segment $[\bar{x}, f\bar{x}]$ has at least two points of C.

With this additional hypothesis f has a fixed point.

For instance, by Theorem 9 there is an $x_0 \in C$ such that $||x_0 - fx_0|| = d(fx_0, C)$.

Let $x_0 \neq fx_0$. Set $y = \lambda x_0 + (1-\lambda) fx_0$, $0 < \lambda < 1$. Then

$y_0 - x> \leq 0$ for all $x \in C$. If $x = fy_0$ then we get that $y_0 = fy_0$.

If C is not a weakly compact set, then the following statement gives a result similar to [7].

Theorem 6. *Let C be a nonempty convex subset of a Hilbert space H and $f : C \to H$ continuous monotone map. If C has a weakly compact, convex subset C_0 such that the set*

$$B = \{y \in C : < fy, y - x> \leq 0 \text{ for all } x \in C_0\}$$

is weaklky compact, then there is a $y \in C$ such that

$$<fy_0, y_0 - x> \leq 0 \text{ for all } x \in C.$$

For the following terminologies, see Mehta [11].

As regards the preference correspondence P one assigns for each bundle x in the consumption set, $P(x)$ is regarded as the set of all bundles that are strictly preferred to x, P is irreflexive binary relation. P has a maximal element x_0 if Px_0 is empty, i.e. if there is no bundle strictly preferred to x_0.

In the end, a result for maximal elements in Mathematical Economic is given. Mehta [11] has discussed results of this nature and has shown the existence of maximal elements for an economics agent.

Theorem 7. *Let $P : C \to 2^C$ be a preference correspondence where C is a nonempty closed convex subset of a Hausdorff topological vector space E. Suppose that*

(i) $x \notin Px$ *for each* $x \in C$,

(ii) Px *is closed for each* $x \in C$,

(iii) $P^{-1} y = \{x \in C : y \in Px\}$ *is convex for each* $y \in C$,

(iv) *assume that C is covered by finitely many closed sets.*

Then there is a maximal element for P.

Proof. Suppose there is no maximal element so

$$P(x) \neq \phi \text{ for all } x \in C.$$

Define $G : C \to 2^C$ by $Gx = C \setminus Px$. Then G is an open valued map for each $x \in C$.

Since $C = \bigcup_1^n Px_i$, so $\bigcap_1^n Gx_i = \bigcap_1^n (C / Px_i) = \cup_1^n Px_i^c = \phi$.

So G is not a *KKM*-map. Recall that if G is an open valued *KKM*-map then the family $\{Gx : x \in C\}$ has the finite intersection property.

Therefore $z = \sum_1^n \alpha_i x_i \notin \bigcup_1^n Gx_i$, and $z \in \bigcap_1^n Px_i$.

Thus $x_i P^{-1}z$. But $P^{-1}z$ is convex so $z = \sum_1^n \alpha_i x_i \in P^{-1}z$, *i.e.*,

[3] F.E. Browder, Fixed point theorems for noncompact mappings in Hilbert space, *Proc. Nat. Acad. Sci.*, U.S.A. **53** (1965), 1272-1279.

[4] F.E. Browder. Fixed point theory of multivalued mappings in topological vector spaces, *Math. Annalen* **177** (1968), 283-301.

[5] Ky Fan, A Generalization of Tychonoff's fixed point theorem. *Math. Annalen.* **142** (1961), 305-310.

[6] Ky Fan, Extensions of two fixed point theorems of F.E. Browder. *Math. Z.*, **112** (1969), 234-240.

[7] P. Hartman and G. Stampacchia, On some nonlinear elliptic differential equations. *Acta Math.*, **115** (1966), 271-310.

[8] W.K. Kim, Some applications of the Kakutani fixed point theorem. *Math. Anal. Appl.*, **131** j(1987), 119-122.

[9] T.C. Lin, Convex sets, fixed points variational and minimax inequalities. *Bull. Austr. Math. Soc.*, **34** (1986), 107-117.

[10] T.C. Lin, Fan's Lemma, fixed points and maximal elements for noncompact sets, (preprint).

[11] G. Mehta, Fixed points, equilibria and maximal elements in linear topological spaces. *Comment. Math. Univ. Carolinae,* **28** (1987), 377-385.

[12] S.P. Singh, B. Watson and P. Srivastava, *Fixed Point Theorems, Best Appximation and the KKM-Map Principle*, Kluwer Acad. Publishers, 1997.

[13] H. Sonnenschein, Demand theory without transitive preference with applications to the theory of competitive equilibrium. *Preference, Utility and Demand.* J. Chipman et al. Eds, Harcourt, Brace and Jovanovich, New York, 1971, 107-117.

[14] A. Villar, Operator theorems with applications to distributive problems and equilibrium models. Lecture Notes in Economics and Math. Systems, Springer-Verlag, 1992.

Jñānābha, Vol. 30, 2000

SOME EXPECTATIONS ASSOCIATED WITH MULTIVARIATE GAMMA AND BETA DISTRIBUTIONS INVOLVING THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

By

R.C. Singh Chandel and P.K. Vishwakarma
Department of Mathematics, D.V. Postgraduate College,
Orai-285001, U.P., India

(Received : December 31, 1997)

ABSTRACT

In the present paper, we obtain some density functions associated with the multivariate gamma and beta distributions and make their applications to obtain the expectations involving multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha [51], p.64). Finally. we also derive the moments for these multivariate beta and gamma distributions and discuss their special cases.

1. Introduction. Different distributions have discussed by various authors Block and Rao [1], Carlson [2], Daley [5] Datt [6]. Kabe [8], Kaufman, Mathai and Saxena [9], Kendall [10], Khatri and Pillai [11,12]. Khatri and Srivastava [13], Littler and Fackerell [15], Lukacs and Naha [16], Lukacs [17], Mathai ([18] to [29]), Mathai and Rathie ([30] to [35]), Mathai and Saxena ([36] to [42]), Miller [43], Pillai, A1-Ani and Jouris [44], Pillai and Jouris [45], Pillai and Nagarsenker [46], Robbins and Pitman [47], Strawderman [52], Thaung [53], and Wilks [54]. Srivastava and Singhal [50], studied many of the classical statistical distributions, which were associated with the beta and gamma distributions. Further Exton [7], discussed generalized beta and gamma distributions with other special multivariate distributions,like Dirichlet distributions and multivariate normal distributions. He also discussed the expectations of some functions invoiving Lauricalla's multiple hypergeometric functions [14]. Recently the authors Chandel and Vishwakarma [4] discussed some multivariate beta and gamma distributions and made their applications to derive their expectations in terms of different multiple hypergermetric functions of several veriables.

* *Present Address . HA-14 Staff Colony, Narsingarh (Birlapur), Damoh,M.P. ,India*

In the present paper, we extend the above work and establish some probability density functions associated with the multivariate beta and gamma distributions and make their applications to obtain some expectations involving the most generalized multiple hypergeometric function of Srivastava and Daoust [48] (see also Srivastava and Manocha ([51],p.64). Finally,we also drive the moments for these multivariate beta and gamma distributions and discuss their special cases.

2. Formulae Required. For ready stock, in this section we write the following results which will be used in our investigations:

The Liouville's Theorem (Also see Chandel [3,p.83 (3.1)]

$$(2.1)\quad \int_0^\infty \ldots \int_0^\infty f(x_1+\ldots+x_n) \quad x_1^{\mu_1-1}\ldots x_n^{\mu_n-1}\, dx_1\ldots dx_n$$

$$= \frac{\Gamma(\mu_1)\ldots\Gamma(\mu_n)}{\Gamma(\mu_1+\ldots+\mu_n)} \int_0^\infty f(t) \quad t^{\mu_1+\ldots+\mu_{n-1}}\, dt,$$

provided that $Re\,(\mu_i) > 0,\ i=1,\ldots,n$.

Euler's definition for gamma function

$$(2.2)\quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad Re\,(z) > 0.$$

The definition of beta function (see, Srivastava and Manocha [51,p.26 eq.(4.6)])

$$(2.3)\quad B(\alpha,\beta) = \int_0^\infty \frac{\mu^{\alpha-1}}{(1+\mu)^{\alpha+\beta}}\, d\mu,\ Re\,(\alpha) > 0,\ Re\,(\beta) > 0.$$

3. Multivariate Gamma Distribution. Consider the function

$$(3.1)\ f(x_1,\ldots,x_n) = \frac{\Gamma(\mu_1+\ldots+\mu_n)\,\lambda^{\mu+\mu_1+\ldots+\mu_n}}{\Gamma(\mu_1)\ldots\Gamma(\mu_n)\quad \Gamma(\mu+\mu_1+\ldots+\mu_n)} \qquad exp\{-(x_1+\ldots+x_n)\lambda\}$$
$$(x_1+\ldots+x_n)^{\mu}$$

provided that $Re(\lambda) > 0,\ x_i \geq 0,\ Re\,(\mu_i) > 0,\ i=1,\ldots,n$ and $f(x_1,\ldots,x_n) = 0$ elsewhere.

Making an appeal to (2.1) and (2.2) the value of multiple integral of $f(x_1,\ldots,x_n)$ over the region defined above in (3.1) becomes unity.

Hence $f(x_1,\ldots,x_n)$ is a probability density function for multivariate gamma distribution.

4. Expectation Associated with Multivariate Gamma Distribution.

The expectation value of the function $g(x_1,\ldots,x_n)$is defined as

$$(4.1) < g(x_1,\ldots,x_n) > = \int_\infty^\infty \ldots \int_\infty^\infty f(x_1,\ldots,x_n)\, g(x_1,\ldots,x_n) dx_1\ldots dx_n$$

corresponding to density function $f(x_1,\ldots,x_n)$defined by (3.1).

Consider the function

$$(4.2)\ g_1(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}];[(b'),\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}];[(d'),\delta'];\dots;\end{matrix}\ \begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1(x_1+\dots+x_n)^{\nu_1},\dots,z_n(x_1+\dots+x_n)^{\nu_n}\right]$$

where $F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}$ is most generalized multiple hypergeometric function of Srivastava and Daoust [48] (also see Srivastava and Manocha [51, (18),(19),(20),p.64)]).

Now making an appeal to (2.1) and (2.2)the expectation of $g_1(x_1,\dots,x_n)$having density function $f(x_1,\dots,x_n)$ is given by

$$(4.3)\ \langle g_1(x_1,\dots,x_n)\rangle = F^{A+1:B';\dots;B^{(n)}}_{C\ :D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}];\\ [(c):\Psi',\dots,\Psi^{(n)}];\end{matrix}\ \begin{matrix}[\mu+\mu_1+\dots+\mu_n:\nu_1,\dots,\nu_n]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1}{\lambda_1^{\nu_1}},\dots,\frac{z_n}{\lambda_n^{\nu_n}}\right]$$

provided that

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\nu_i>0,\ i=1,\dots,n.$$

Corresponding to density fuction $f(x_1,\dots,x_n)$ defined by (3.1), if we consider the function

$$(4.4)\ \langle g_2(x_1,\dots,x_n) = F^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\\ [(c):\Psi',\dots,\Psi^{(n)}]:[(d'):\delta'];\end{matrix}\ \begin{matrix}\dots;[(b^n):\phi^n];\\ \dots;[(d^n):\delta^n];\end{matrix}\ z_1x_1^{\alpha_1}(x_1+\dots+x_n)^{\nu_1},\dots,z_nx_n^{\alpha_n}(x_1+\dots+x_n)^{\nu_n}\right]$$

Then expectation of g_2 is given by

$$(4.5)\ \ g_2(x_1,\dots,x_n)\rangle = F^{A+1:B';\dots;B^{(n)}+1}_{C+1:\ D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],\\ [(c):\Psi',\dots,\Psi^{(n)}],\end{matrix}\ \begin{matrix}[\mu+\mu_1+\dots+\mu_n:\alpha_1+\nu_1,\dots,\alpha_n+\nu_n]:[(b'):\phi'],[\mu_1:\alpha_1];\dots;[(b^n):\phi^{(n)}],[\mu_n:\alpha_n];\\ [\mu+\mu_1+\dots+\mu_n:\alpha_1,\dots,\alpha_n]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\ z_1/\lambda^{\alpha_1+\nu_1},\dots,z_n/\lambda^{\alpha_n+\nu_n}\right],$$

valid if

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}-\alpha_i-\nu_i>0,\ i=1,\dots,n.$$

5. The Multivariate Beta Distribution.

Consider the function $F(x_1,\dots,x_n)$defined by

$$(5.1)\ F(x_1,\dots,x_n) = \frac{\Gamma(\mu_1+\dots+\mu_n)\,\Gamma\alpha+\lambda+\mu_1+\dots+\mu_n)\,(x_1+\dots+x_n)^{\alpha}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\,\Gamma(\lambda)\,\Gamma\,(\alpha+\mu_1+\dots+\mu_n)\,(1+x_1+\dots+x_n)^{\alpha+\lambda+\mu_1+\dots+\mu_n}}$$

$Re(\alpha)>0, Re(\mu_i)>0, x_i>0\ i=1,\dots,n$ and $F(x_1,\dots,x_n)=0$ elsewhere.

Now making an appeal to (2.1) and (2.3) the value of multiple integral of $F(x_1,\dots,x_n)$ over the region defined above in (5.1), becomes unity.

Hence $F(x_1,\dots,x_n)$ is probability density function for multivariate beta distribution.

6. Expectation Associated with Multivariate Beta Distribution.

Correspondiing to density function $f(x_1,\dots,x_n)$ defined by (5.1), consider the function

$$(6.1)\ G\,(x_1,\dots,x_n) = F^{A':B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(b'):\phi'];\dots;\\ [(c):\psi',\dots,\psi^{(n)}]:[(d'):\delta'];\dots;\end{matrix}$$

$$\begin{matrix}[(b^{(n)}):\phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\ \frac{z_1 x_1^{\xi_1}(x_1+\dots+x_n)^{\eta_1}}{(1+x_1+\dots+x_n)^{\zeta_1+n_1}},\dots,\frac{z_n x_n^{\xi_n}(x_1+\dots+x_n)^{\eta_n}}{(1+x_1+\dots+x_n)^{\zeta_n+\eta_n}}$$

Then expectation of $G(x_1,..,x_n)$ is given by

$$(6.2)\ \langle G(x_1,\dots,x_n)\rangle$$

$$= F^{A+1:B'+;\dots;B^{(n)}+1}_{C+2:D';\dots;D^{(n)}}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}],[\alpha+\mu_1+\dots+\mu_n:\eta_1+\xi_1,\dots,\eta_n+\xi_n]:\\ [(c):\psi',\dots,\psi^{(n)}],[\mu_1+\dots+\mu_n:\xi_1,\dots,\xi_n],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\phi'],[\mu_1:\xi_1],\dots,[(b^{(n)}):\phi^{(n)}],[\mu_n:\xi_n];\\ [\alpha+\lambda+\mu_1+\dots+\mu_n:\eta_1+\zeta_1,\dots,\eta_n+\zeta_n]:[(d'):\delta'];\dots;[(d^n):\delta^n];\end{matrix}\ z_1,\dots,z_n\right]$$

where

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\phi_j^{(i)}+\zeta_i-\xi_i>0,$$

$i=1,\dots,n.$

7. Moment Generating Function (*m.g.f.*)for Gamma Distribution.

The *m.g.f.* is defined as

$$(7.1)\quad M(t_1,\dots,t_n)\quad \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} e^{x_1t_1+\dots+x_nt_n} f(x_1,\dots,x_n)\,dx_1\dots dx_n,$$

provided that the integral is a function of the parameters $t_1,\dots,t_n$ only. Thus *m.g.f.* for multivariate gamma distribution (3.1) is given by

$$(7.2)\ M(t_1,\dots,t_n) = \frac{\Gamma(\mu_1+\dots+\mu_n)\ \lambda^{\mu+\mu_1+\dots+\mu_n}}{\Gamma(\mu_1)\dots\Gamma(\mu_n)\ \Gamma\,(\mu+\mu_1+\dots+\mu_n)}$$

$$\int_0^{\infty}\dots\int_0^{\infty} e^{\,x_1t_1+\dots+x_nt_n}\,e^{-(x_1+\dots+x_n)\lambda}\,(x_1+\dots+x_n)^{\mu}\,x_1^{\mu_1-1}\dots x_n^{\mu_n-1}\,dx_1\dots dx_n$$

Now making an appeal to (2.1) and the result due to Srivastava[49,p,4.(12).]

$$\sum_{m_1,\ldots,m_n=0}^{\infty} f(m_1+\ldots+m_n)\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{M=0}^{\infty} f(M)\ (x_1+\ldots+x_n)^M, n \geq 1,$$

we finally derive

(7.3) $M(t_1,..,t_n) = F_D^{(n)}(\mu+\mu_1+\ldots+\mu_n, \mu_1,\ldots,\mu_n; \mu_1+\ldots+\mu_n; \frac{t_1}{\lambda},\ldots,\frac{t_n}{\lambda})$,

where $F_D^{(n)}$ is Lauricella's fourth multiple hypergeometric function of several variables [14].

As a special case for $\mu = 0$, (7.3) gives

(7.4) $M(t_1,\ldots,t_n) = \prod_{i=1}^{n} (1-t_i/\lambda)^{-\mu_i}$.

8. Moments for Gamma Distribution. The moment $\mu'_{r_1,\ldots,r_n}$ for gamma distribution about $(0,\ldots,0)$ of order $r_1,\ldots,r_n$, is defined as the coefficient of $\frac{t_1^{r_1}}{r_1!}\cdots\frac{t_n^{r_n}}{r_n!}$ in $M(t_1,\ldots,t_n)$ when it is expended in powers of $t_1,\ldots,t_n$. Thus an appeal to (2.1) gives

(8.1) $\mu'_{r_1,\ldots,r_n} \dfrac{(\mu_1)_{r_1}\cdots(\mu_n)_{r_n} \quad (\mu+\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}}{(\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n} \quad \lambda^{r_1+\ldots+r_n}}$.

9. Moment for Beta Distribution. The moment $\mu'_{r_1,\ldots,r_n}$ of density function $F(x_1,\ldots,x_n)$ about $(0,\ldots,0)$ for beta distribution is defined as

(9.1) $\mu'_{r_1,\ldots,r_n} = \int_{\infty}^{\infty}\cdots\int_{\infty}^{\infty} x_1^{r_1}\ldots x_n^{r_n} F(x_1,..,x_n)\ dx_1\ldots dx_n$.

Now substituting the value of $F(x_1,\ldots,x_n)$ from (5.1) in (9.1) and making an appeal to (2.1) and (2.3), we finally derive

(9.2) $\mu'_{r_1,\ldots,r_n} = \dfrac{\Gamma(\alpha+\lambda-(r_1+,\ldots,+r_n)\ (\mu_1)_{r_1}\cdots(\mu_n)_{r_n}}{\Gamma(\lambda)\ \Gamma(\alpha+\mu_1+\ldots+\mu_n)\ (\mu_1+\ldots+\mu_n)_{r_1+\ldots+r_n}}$.

10. Special Cases. For $n=1$, from (7.3), we derive the following *m.g.f.* for gamma distribution:

(10.1) $M(t_1) = (1-t_1/\lambda)^{-\mu+\mu_1}$.

For $n=1$, from (8.1), we obtain the following moment of r_1th order about origin for gamma distribution:

(10.2) $\mu'_{r_1} = \dfrac{(\mu+\mu_1)_{r_1}}{\lambda^{r_1}}$.

Also for $n=1$, (9.2) gives the following moment of r_1th order for beta distribution :

(10.3) $\mu'_{r_1} = \dfrac{\Gamma(\alpha+\lambda-r_1)}{\Gamma(\lambda)\,\Gamma(\alpha+\mu_1)}$.

ACKNOWLEDGEMENT

The authoirs are thankful to the Council of Science and Technology, Uttar Pradesh, India for providing financial assistance to this project CST/D-196/Math/27/86.

REFERENCES

[1] H.W. Block and B.R. Rao, A beta warning-time distribution and distended beta distribution,*Sankhya Ser. B*, **35** (1973), 79-84.

[2] B.C. Carlson, Integrandes a Deux forms quadratiques, *C.R. Acad.Sci.Paris*, **274** (1972),1458-1461.

[3] R.C.S. Chandel, The product of certain classical polynomials and the generalized Laplacian operator,*Ganita*, **20** (1969,79-87.

[4] R.C.S. Chandel and P.K. Vishwakarma, Some probability distributions and expectations associated with beta and gamma distributions involving multiple hypergeometric functions of several variables, *Jñānābha*, **27** (1997), 131-137.

[5] D. J. Daley, Computation of bivariate and trivariate normal distributions, *J. Roy. Statist, Soc. Ser. C*, **23** (1974), 435-438.

[6] J. E. Datt, On computing probability integral of a general multivariate *t*. *Biometrica*, **62** (1975), 201-208.

[7] H. Exton, *Multiple Hypergeometric Functions and Applications*, John Wiley and Sons, Inc., New York, London, Sydney, Toronto, 1976.

[8] D.G. Kabe, On an exact distribution of a Class of multivariate test criteria, *Ann. Math. Statist.*, **23** (1962), 1197-1200.

[9] H. Kaufman, A.M. Mathai and R.K. Saxena, Distributions of randam variables with randam parameters, *South Afr. Statist. J.*, **3** (1969), 1-7.

[10] M.G. Kendall, *Advance Theory of Statistics*, Vol.2, Griffin, 1951.

[11] C.G. Khatri and K.C.S. Pillai, Some results on non-central multivariable beta distribution and moments of traces of two matrices, *Ann.Math. Statist.*, **36** (1965),1511-1520.

[12] C.G. Khattri and K.C.S. Pillai, On the non-central distributions of two criteria in multivariate analysis of variance, *Ann.Math. Statist.*, **39** (1968), 215-216.

[13] C.G. Khatri and M.S. Srivastava, On exact non-null distributions of likelihood ratio criterian for sphericity test and equality of two covariance matrices, *Sankhya,Ser, A*, **33** (1971), 201-206.

[14] G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend. Circ., Mat. Palermo*, **7** (1893), 111-158.

[15] R.A. Littler and E.D. Fackerell, Transition densities for neutral multi-allelediffusion models, *Biometrika*, **31** (1975), 117-123.

[16] E. Lukacs and R.G. Naha, *Applications of Characteristic Functions*, Griffin (1963).

[17] E. Lukacs, *Characteristic Function*, Griffin, 1970.

[18] A.M. Mathai, *Applications of Generalized Special Functions, Monograph*, Indian Statistical Institute and McGill University, 1970.

[19] A.M. Mathai, The exact distribution of a criterion for testing the hypothesis that several populations are indentical, *J.Indian Statistical Assoc.* **8**(1970),1-17.

[20] A.M. Mathai, The exact distribution of Bartlett's criterion for testing equality of covariance matrices, *Publ. L'I Sup, Paris*, **19** (1970),1-15.

[21] A.M. Mathai, Statistical Theory of distribution and Meijer's *G*-function, *Metron*,

28 (1970), 122-146.

[22] A.M. Mathai, A representation of an H-Function suitable for practical applications, *Indian Statistical Institute, Calcutta, Technical Report, Math. Statist., 29-70.*

[23] A.M. Mathai, On the distribution of the likelihood raatio criterion for testing linear hypothesis on regression coefficients, *Ann. Inst. Statist. Math.*, **23** (1971),181-197.

[24] A.M. Mathai, An expansion of Meijer's G-function and the distribution of product of independent beta variates, *S.Afr. Statist. J.*,**5** (1971), 71-90.

[25] A.M. Mathai, The exact non-null distributions of a collection of multivariate test statistics, *Publ. L'I SUP, Paris.* **20** No.1(1971).

[26] A.M. Mathai, The exact distributions of three criterion associated with Wilk's concept of generalized variance, *Sankhya, Ser. A* **34** (1972), 161-170.

[27] A.M. Mathai, The exact non-central distribution of the generalized variance, *Ann. Inst. Statist. Math.*, **24** (1972), 53-65.

[28] A.M. Mathai, The exact distribution of a criterion for testing that the covariance matrix in diagonal, *Trab. Estadistica*, **28** (1972),111-124.

[29] A.M. Mathai, A few remarks on the exact distributions of likelihood ratio riteria - *I, Ann. Inst. Statist. Math.* **24** (1972).

[30] A.M. Mathai and P.N. Rathie, An expansion of Meijer's G- function and its applications to statistical distributions, *Acad. Roy. Belg., Ci Sci.* (5) **56** (1970), 1073-1084.

[31] A.M. Mathai and P.N. Rathie, The eaxact distribution of Votaw's criterion, *Ann. Inst. Statist. Math.*, **22** (1970), 89-116.

[32] A.M. Mathai and P.N. Rathie, The eaxact distribution for the sphericity test, *J. Statist. Res. (Dacca)*, **4** (1970), 140-159.

[33] A.M. Mathai and P.N. Rathie, The eaxact distribution of Wilks criterion, *Ann. Math. Statist.* **42** (1971), 1010-1019.

[34] A.M. Mathai and P.N. Rathie, The eaxact distribution of Wilks generalized variance in the non-central linear case, *Sankhya, Ser. A*, **33** (1971), 45-60.

[35] A.M. Mathai and P.N. Rathie, The problem of testing independence, *Statistica*, **31** (1971), 673-688.

[36] A.M. Mathai and R.K. Saxena, On a generalized hypergeometric distributicn *Metrika*, **11** (1966), 127-132.

[37] A.M. Mathai and R.K. Saxena, Distribution of a product and the structural setup of densities, *Ann. Math. Ststist.* **40** (1969), 1439-1448.

[38] A.M. Mathai and R.K. Saxena, Applications of Special Functions in the characterization of probability distributions, *S.Afr. Statist. J.*, **3** (1969), 27-34.

[39] A.M. Mathai and R.K. Saxena, Extension of Euler's integrals through statistical techniques, *Math. Nachr.*, **51** (1971), 1-10.

[40] A.M. Mathai and R.K. Saxena, A generalized probability distribution, *Univ. Nac. Tucaman Rev. Ser.* A, **21** (1972), 193-202.

[41] A.M. Mathai and R.K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Monograph, Department of Mathematics, Mc Gill University, July 1972.

[42] A.M. Mathai and R.K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences*, Springer-Verlag, Berlin, Heidelberg, New York, 1973

[43] K.S. Miller, *Multidimensional Gaussian Distribution*, Wiley, New York, 1964.

[44] K.C.S. Pillai, S. Al-Ani, and G.M. Jouris, On the distributions of the roots of a

covariance matrix and Wilks criterion for tests of three hypothesises, *Ann. Math. Statist.*, **40** (1969), 2033-2040.

[45] K.C.S. Pillai, and G.M. Jouris,Some distribution problems in multivariate complex Gaussian case, *Ann, Math. Statist.*, **42** (1971), 517-525.

[46] K.C.S. Pillai, and B.N. Nagarsenker, On the distribution of the sphericity test criterion in classical and complex normal populations having unknown covariance matrices, *Ann. Math. Statist.*, **42** (1971), 764-767.

[47] H.Robbins and E.J.G. Pitman, Applications of the method of mixture toFadratic forms in normal variances, *Ann. Math. Statist.*, **20** (1949), 318-324.

[48] H.M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser.* A 72. = *Indag. Math.*, **31** (1969), 449-457.

[49] H.M. Srivastava, Certain double integrals involving hypergeometric functions, *Jñānābha Sect. A*, **1** (1971), 1-10.

[50] H.M. Srivastava and J.P. Singhal, On a class of generalized hypergeometric distritutions, *Jñānābha Sect. A*, **2** (1972), 1-9.

[51] H.M. Srivastava and H.L.Manocha, *A Treatise on GeneratingFunctions*, John Wiley and Sons Inc. New York, 1984.

[52] W.E. Strawderman, Minimax estimation of location parameters for certain spherically symmetrical distributions, *J. Multivariate Anal.*, **4** (1974), 255-264.

[53] L. Thaung, Exponential family distribution with a truncation parameter, *Biometrika*, **62** (1975), 218-220.

[54] S.S. Wilks, *Mathematical Statistics*, Wiliy, New York, 1962.

Jñānābha, Vol. 30, 2000

MATHEMATICAL MODELS ON EPIDEMIOLOGICAL PROBLEMS WITH SPECIAL REFERENCE TO AIDS AND ITS REDUCTION STRATEGIES - USE OF COMPOUND BINOMIAL DISTRIBUTION.

By

S.S. Mishra, D.N. Pandey and A.S. Mishra

Department of Mathematics and Statistics, Dr. R.M.L. Avadh University, Faizabad-224001, U.P., India

(Received : February 15, 1999)

ABSTRACT

In this paper, an attempt has been made to make use of compound binomial distribution in dealing with problems of AIDS. Among methods include interval estimation, appoximation through Gaussian distribution and M.L.E. method etc. Finally, graphical approach has been used to demonstrate the comparative study of incidence infection rate corresponding to various sources of infection.

1. Introduction. When an outcome of the phenomenon is surrounded by uncertainties, it seems to be much difficult to quantify the amount of an outcome of that phenomenon. In order to effectively deal with probalilistic uncertainty, probability distribution theory has been developped to solve the above problem. Broadly categorised in two types-discret distribution and continuous distribution theories have tremendous potential for application in various wide ranging areas such as physical sciences, life sciences, epidemiology, management business, commerce, psychology and sociology etc. In 1974, Bachev and Petkova [1] have presented a mathematico-statistical analysis of the cycle recurrence in the droplet infections of communicable diseases. They have used dispersion analysis in the study of cycle recurrence or cyclicity.

In 1978, similarly, Bender [2] has conceptualised mathematical modelling in the areas of sex preference and sex ratio and problem of choices etc. Nevertheless, he has also studied Monte Carlo simulation technique which are applicable to a doctor's waiting room and sediment volume etc. Probabilistic model has also been used by him in the study of radio active decay, optimal facility location and distribution of particle size etc. Now a days, epidemiological problems have seriously attracted the attention of scientists involved therein. Different types of problems connected with epidemiology have been modelled mathematically and

statistically.

In 1989, Edwards and Hamson [3] have also presented mathematical (statistical) modelling based on random variables. They have suggested various discrete and continuous random variables and their corresponding distributions to model different scientific and real life situations. They have especially discussed binomial, Poisson, exponential and normal distributions and their applications to various real life situations. Their approach to statistical modelling based on random varibles has undoubtedly added a new dimension to its study. Further more, in 1994 Kapur [5] has also described various aspects of mathematical modelling and its application in differnt areas such as physical and life sciences including epidemiology.

Under the category of epidemiological problems, recently, it is none but a problem of AIDS which has shaken the root of developed and developing societies and consequently gripped so badly that it has posed a great challenge before the epidemiologists. Unless, we effectively quantify the seriousness of disease, we cannot adopt any efficient reduction and prevention strategies to contain the HIV infection. Various mathematical modellings carried out previously could not have been able to precisely revealed such a fact which could be instrumental to adopting any effective reduction strategy. Recently Kotia and Srivastava [4] have made an effort to analyse impact of various sources of infection in order to seriously understand the problems of AIDS, so that infection rate could be contained effectively. Their study has made a hallmark in the investigations of problems of AIDS based on various sources of infection in the Indian perspective. Of late, Flanders, W.D. and Kleinbaum, D.G. [6] have presented a study on basic models for disease occurrence in Epidemiology. They have analysed disease occurrence with special reference to smoking deaths due to lung cancer in UK. More recently, Misra [7] has made an attempt in analysing the seriousness of AIDS infection by making use of Poisson distribution. However, his analytical approach could remain quite theoretical and couldnot reveal the process of data-fit without which purpose of investigation could not be fulfilled completely.

In the present paper an effort has been made by us to make use of compound binomial distribution in dealing with problems of AIDS. Here, it is a natural inquisitiveness that if any infected person of HIV transmits his or her HIV in several other persons with certain amount of probability along with the probability with which an individual infection is developing then what will be the probability or probabilitic

rate that how many persons with how much probability would be infected in future out of total number of infected persons due to an individual HIV positive patient. Here we have been contemplating this situation of AIDS infection by making use of compound binomial distribution. Here the present paper is exclusively devoted to exhibit various analytical aspects such as theoretical, numerical and graphical in order to reach some index as a probabilistic rate of occurrence of AIDS infection which may be subsequently found helpful in formulating reduction as well as prevention strategies. Among methods include interval estimation, approximation through Gaussian distribution and MLE method etc. Finally, graphical aproach has been used to demonstrate the comparative study of incidence infection rate corresponding to various sources of infection.

2. Assumptions and Preliminaries. Before we use the compound binomial distribution in statistically analysing the problems of AIDS we have some underlined assumptions under which bionomial distribution can be applied in the afore said analysis :-

(i) AIDS disease can Independently occur among the group of people.

(ii) Risk of desease is equal with the change of time.

(iii) The occurrence of disease is rare and random in its beginning but subsequently it can cover the globly for the given geographical region.

Here we define compound binomial distribution as follows :

Let us suppose that $X_1, X_2, X_3, \ldots$ are indentically and independently distributed Bernouli variates with $P[X = 0] = p$ and $P[X = 1] = q = 1-p$. For a fixed n, the random variable $X = X_1 + X_2 + X_3 + \ldots + X_n$ is a binomial variate with parameters n and p and probability function :

$$p\,(x = r) = \binom{n}{r} p^r q^{n-r},\ r = 0,1,2, \ldots, n.$$

which gives the probability of r successes in n independent trials with constant probability p of success for each trial.

Now suppose that n, instead of being regarded as a fixed constant, is viewed as a random variable following Poisson law with Parameter λ.Then,

$$P\,(n = k) = \frac{e^{-\lambda}\lambda^k}{k!}\,;\ k = 0,\ 1,2, \ldots$$

In such a case x is said to have compound binomial distribution. The joint probability funciton of x and n is given by

$$P\,(X = r \cap n = k) = P\,(n = k)\,P\,(x = \frac{r}{n} = k) = \frac{e^{-\lambda}\lambda^k}{r} - \binom{k}{r} p^r q^{k-r}.$$

Since $P\,(X = r/n = k)$ is the probability of r successes in k trials. Obviously

$r < k \Rightarrow k > r.$

The marginal distribution of x is given by :

$$P(x = r) = \sum_{k=r}^{\infty} P(X = r \cap n = k)$$

$$= e^{-\lambda} p^r \sum_{k=r}^{\infty} \binom{k}{r} \frac{\lambda^k q^{k-r}}{k!}$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} \sum_{k=r}^{\infty} \frac{(\lambda p)^{k-r}}{(k-r)!}$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{(k-r)!} ; j = k-r.$$

$$= \frac{e^{-\lambda}(\lambda p)^r}{k!} e^{\lambda q},$$

which is a probability function of a Poission Variate with parameter λp. Hence $E(x) = \lambda p$ and variate $(x) = \lambda p$.

Now we estimate λp by making use of maximum likelihood estimation *(MLE)* method these after use interprete it in the term of disease incidence after relating it to the Person-time.

3. Statstical Analysis. Source Wise Appromimation of Probabilities by Gaussian Distribution for 95% Confidence Limit. If *MLE* of λp is C then incidence rate of the disease is represented by *C/PT*. Here it may be of interest to note that if group of cases is smaller the value of probability can be estimated accurately but when group of cases becomes larger then it can be approximated with the help of Gaussian distribution which is shortly demonstrated in the next discussion.

We know that there are several sources of infection through which *HIV* of *AIDS* are spreading in the society. Here we have been made available the statistics corresponding to each and every source of infection from different sero surviellance centres established in various corner of India. This is being displayed by table given as under.

3.1 Table

S.No.	*Sources of Infection*	*No. of Cases*
1.	Sexual Promiscuous	4483
2.	Blood Donors	1682
3.	Recipients of blood	209
4.	Drug users	1647
5.	Others	2247

Source : Ministry of Health, India, 1994

1. **Sexual Promiscuous**

$4363 \leq x \leq 4603$

$0.0194646 < r < 0.0205353$

2. **Blood Donors**

$1602 \leq x \leq 1762$

$0.0158739 \leq r \leq 0.0174593$

3. **Recipients of blood**

$181 \leq x \leq 237$

$0.0133235 \leq r \leq 0.0174457$

4. **Drug Users**

$1567 \leq x \leq 1727$

$0.0190285 \leq r \leq 0.0209969$

5. **Others**

$2157 \leq x \leq 2337$

$0.0159991 \leq r \leq 0.0173342$

3.2 Table

S.No.	Sources of Infection	Rate of Occurrence
1.	Sexual Promiscuous	$0.0194646 \leq r \leq 0.0205353$
2.	Blood Donors	$0.0158739 \leq r \leq 0.0174593$
3.	Recipients of blood	$0.0133235 \leq r \leq 0.0174457$
4.	Drug users	$0.0190285 \leq r \leq 0.0209969$
5.	Others	$0.0159991 \leq r \leq 0.0173342$

4. Graphical Representation.

Here we plot a graph corresponding to each and every source of infection and subsequently we conduct the comparative study of probabilistic occurrence of this disease. From graph we can approximate the following informations :

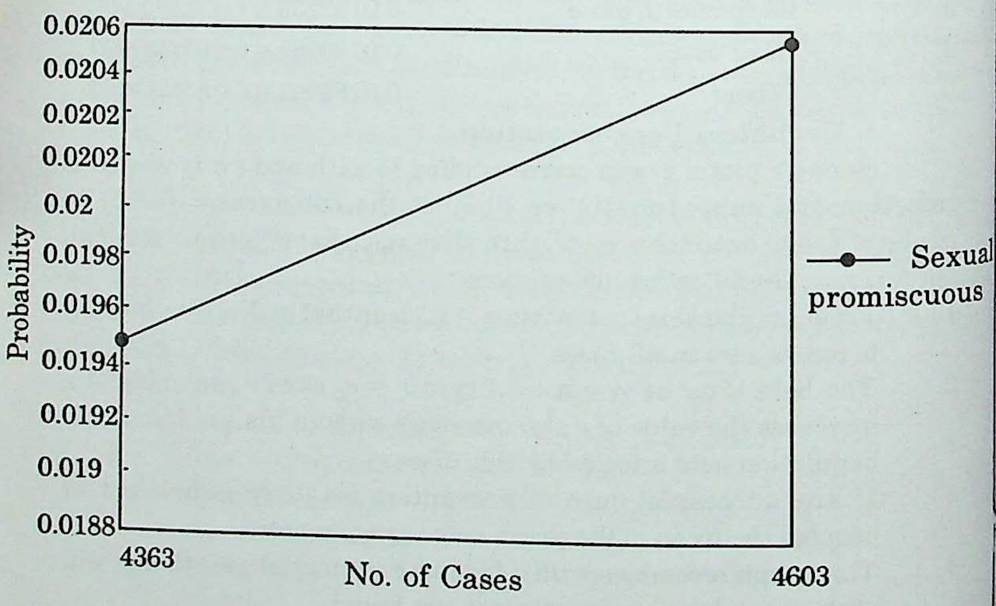
1. From graphical representation it is clear that at $X = 0$ probability becomes zero in all cases.
2. The behaviour of graph is of type $r = r_0\ ekx$ i.e. as value of x increases the value of r also increases without limit unless whole population gets infected by this disease.
3. If any successful cure of prevention srtategy is invoked or adopted the trend of the graph may change reverse.
4. This graph resembles with a famous exponential growth and will continue so long as any curve is not found.

5. Chances of occurrence of disease through recipients of blood source infected are being reflected higher than other sources of infection of a given number of cases.
6. 5% increase in the case will result in an increase of 0.0001 in case of sexual promiscuous, 0.0001 in case of blood donors, 0.0006 in case of recipients of blood, 0.0002 in case of drug users, 0.0001 in case of others.

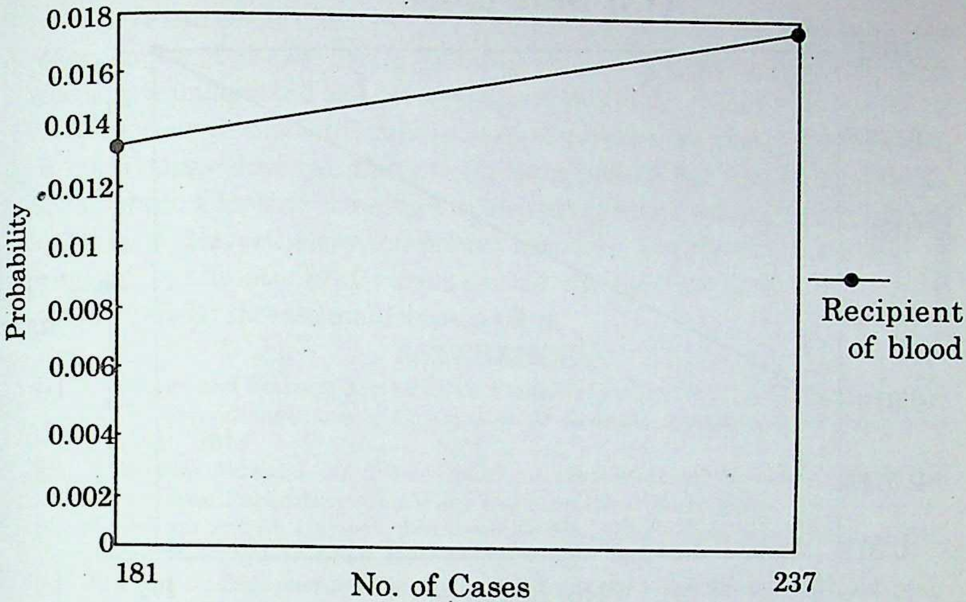
In view of above facts, reduction and prevention strategies may be chalked out efficiently and we may bring a better level of awareness among the people to reduce the infection rate. In the lack any vaccine or cure, we think preventive strategy is only sought cure which can be formulated effectively with the above set of informations.

As we have already mentioned that in case of larger group, probabilities are approximated through Gaussiam distribution with certain degree of confidence limit 95% and we describe the approximation corresponding to each and every source of infection as stated in the above table.

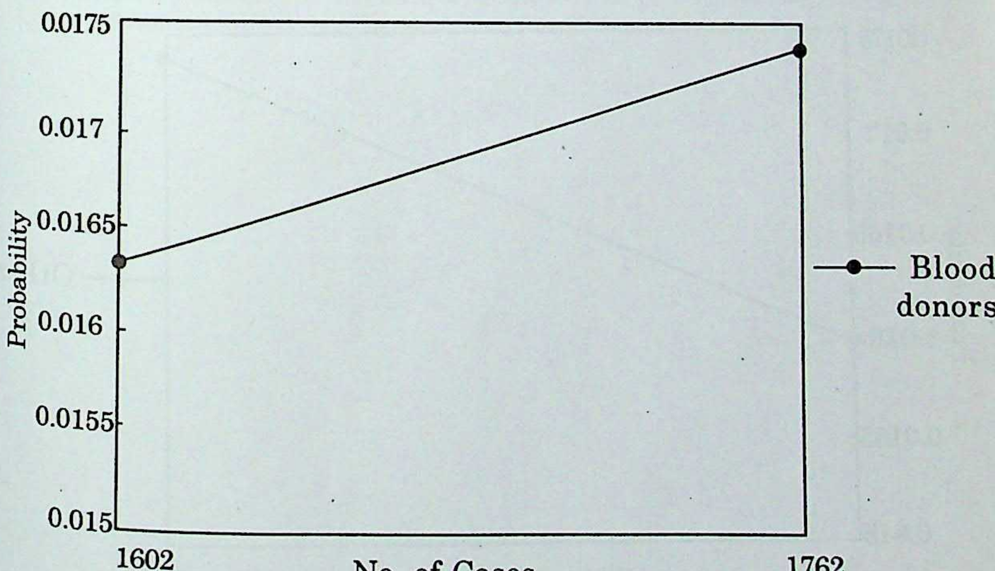
(4.1) Sexual Promiscuous



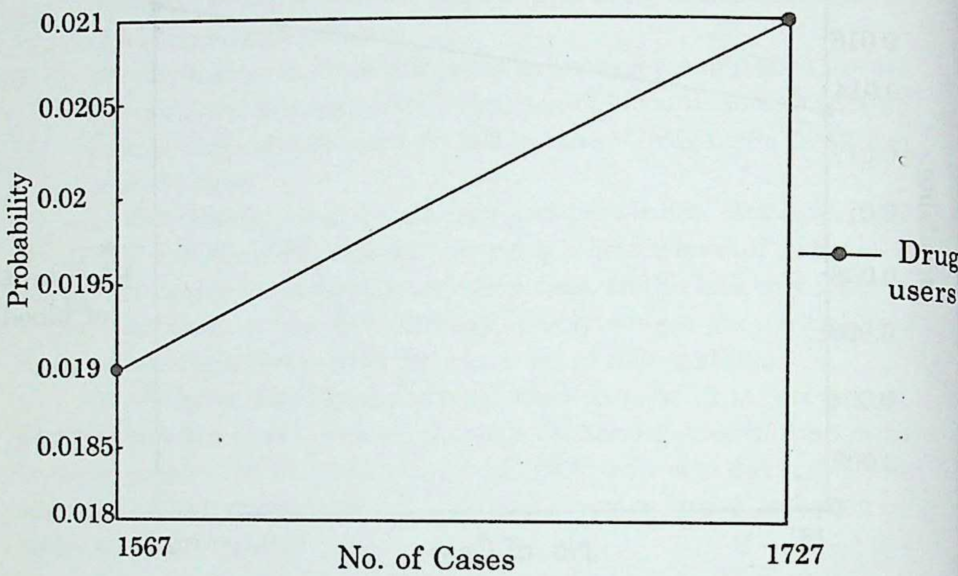
(4.2) Recipient of blood



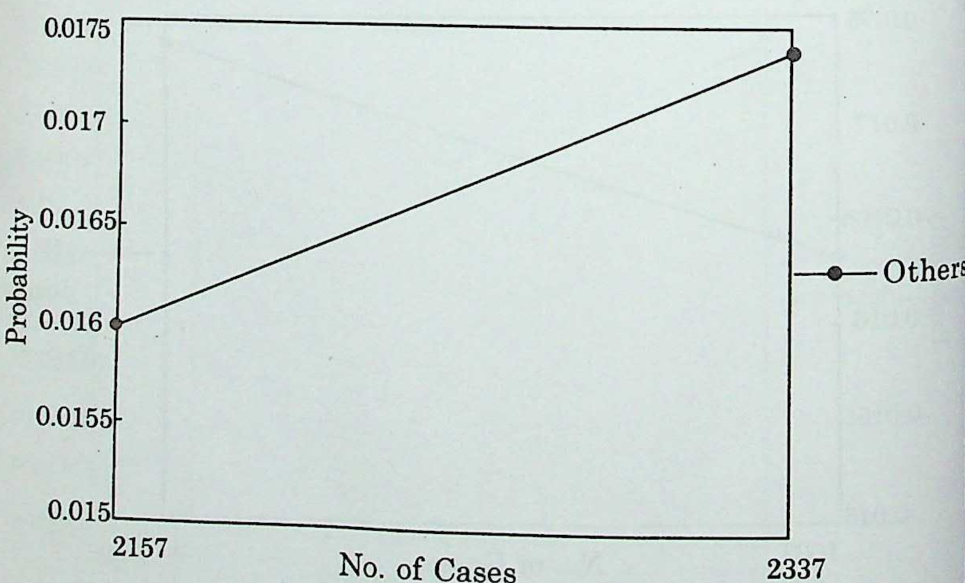
(4.3) Blood donors



(4.4) Drug users



(4.5) Others



5. Conclusion.

We can easily conclude that we have quantified the probabilitic rate of ocurrence of disease (*AIDS*) which is proven as an instrument to make the people understand and get acquainted at which rate *AIDS* is spreading in the society through different sources of infection and how intervariance is taking place dure to different sources of infections. This thing is a sole idea behind adopting the redution and prevention strategies in order to contain it. Nevertheless compound binomial distribution can also be applied in the study of various wide ranging problems as cancer, cardiovasenlar disease and Hepatitis B etc.

REFERENCES

[1] S. Bachev and Petkova, *Mathematico Statistical Analysis of the Cyclic Recurrence in the Droplet Infections of Communicable Diseases*. North Holland Publishing Company 1974.

[2] A. Bender Edward, *An Introduction to Mathematical Modelling*, A Wiley Interscience Publication John Wiley and Sons, New York, 1978.

[3] D. Edward and M. Hamson, *Mathematical Modelling*, Macmillan Education Ltd. Hound Mills, Basing stoke, Hampshire, London, 1989.

[4] J.N. Kapur, *Mathematical Modelling*, H.S. Poplai for Wiley Eastern Limited, 4835/24, Ansari Road Daryaganj, New Delhi, 1994.

[5] A. Kotia and G. Srivastava, *AIDS Diagnosis Treatment and Prevention*, S.K. Jain Publishers and Distributiors 485, Jain Bhawan, B.N. Bhawan, Shahadra, Delhi, 1994.

[6] W. Dana Flanders and G.K. David, Basic Models for disease occurrence in Epidemiology, *International Journal of Epidemiology*, **24 No. 1**, (1995) 1-7.

[7] S.S. Misra (1996), An estimation of disease occurrence in Epidemilogy with special refefrence to *AIDS*, *Vij. Par. Anu. Pat.*, **393** (1996), 161-163.

Jñānābha, Vol. 30, 2000

CLASS OF CERTAIN UNIVALENT FUNCTIONS

By

K.K. Dixit and A.L. Pathak

Department of Mathematics,

Janta College, Bakewar (Etawah)-206124, U.P.

(Received : March 2, 1999)

ABSTRACT

Let $D\ (A,B,\ \alpha\ )$ be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit disc $U = \{z : |z| < 1\}$, and satisfying

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)},$$

$$-1 \le B < A \le 1,\ z \in U,\ 0 \le \alpha < 1,$$

where $w(z)$ is regular in U and satisfies $w(0) = 0,\ |w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U. In this paper we obtain coefficient estimates, radius of convexity for $f(z) \in D\ (A,B,\ \alpha\ )$ and some more interesting results for the subclasses of $D\ (A,B,\ \alpha\ )$.

1. Introduction. Let D denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in unit dist $U = \{z : |z| < 1\}$, and satisfying the condition

$$\left| \frac{f'(z)}{g'(z)} - 1 \right| < 1, \qquad z \in U, \tag{1.1}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

Let $D\ (\delta)$ denote the class of function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, regular in U and satisfying the condition

$$\left| \frac{\{f'(z)/g'(z)\} - 1}{\{f'(z)/g'(z)\} + 1} \right| < \delta, \qquad z \in U, \tag{1.2}$$

where $0 < \delta \le 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

In this paper we introduce the class $D\ (A,\ B,\alpha\ )$ of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular in U and satisfying the condition

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w(z)}{1+B\, w(z)}, \tag{1.3}$$

$- 1 \leq B < A \leq 1,\ z \in U,\ 0 \leq \alpha < 1,$

where $w(z)$ is regular in U and satisfies the conditions $w(0) = 0$, $|w(z)| < 1$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular and univalent in U.

With appropriate choices of A and B and α, the class $D(A, B, \alpha)$ reduces to following known important subclasses :

(i) For $A = 1$ and $B = 0$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class D.

(ii) For $A = \delta$ and $B = -\delta$ and $\alpha = 0$, the class $D(A,B,\alpha)$ reduces to the class $D(\delta)$.

In this paper we obtain sharp coefficient estimates and sharp radius of convexity for $f(z) \in D(A,B,\alpha)$. Also, we obtain the sharp result concerning the radius of convexity for the subclasses of $D(A,B,\alpha)$ associated with each of the cases :

(i) $g(z)$ is starlike in U,

(ii) $g(z)$ is convex in U,

(iii) $Re\, g'(z) > 0$ in U.

Recently, Lakshminarashimhan [3] have studied the class $D(\delta)$ and Ratti [4] have studied the class D. Thus our results naturally generalize the corresponding results of Lakshminarasimhan [3] and Ratti [4].

In fact, an attempt has been made to have a unified and detailed study of these classes of univalent functions.

Throughout this paper we assume that $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$ and R_1, R_2, K_1 and L_1 have the same values as in Lemma 2.1.

2. To establish the results of subsquent section we require the following lemmas:

Lemma 2.1. If $p(z) \in P(A,B,\alpha)$, then on $|z| = r < 1$,

$$Re\left\{\frac{z p'(z)}{p(z)}\right\} \geq \begin{cases} -\dfrac{(A-B)(1-\alpha) r}{[1 + \{(A-B)(1-\alpha) + B\}r](1-Br)}, & R_1 \leq R_2 \\[2ex] \dfrac{(A-B)(1-\alpha) + 2B}{(A-B)(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)(1-\alpha)(1-r^2)}, & R_1 \leq R_2 \end{cases}$$

where $R_1 = (L_1/K_1)^{1/2}$, $R_2 = \dfrac{[1 - \{(A-B)(1-\alpha) + B\} r]}{(1-Br)}$,

$L_1 = [1 - \{(A-B)(1-\alpha) + B\}]\,[1 + \{(A-B)(1-\alpha) + B\} r^2]$

and $K_1 = (1-B)(1+Br^2)$.

The result is sharp. The above lemma follows from Theorem 1 of Anh

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right]=R_1 \qquad \text{at } z=-r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 3) turn out to be special case by taking $A=1$, $B=0$ and $\alpha=0$.

Corollary. If $f(z)\in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then $f(z)$ is convex in

$$|z|<\begin{cases} r_3 & \text{for } R_1\le R_2 \\ r_4 & \text{for } R_2\le R_1 \end{cases}$$

where r_3 and r_4 respectively the smallest positive roots of the following equations:

$$(1-r)\,[1-\{(A-B)(1-\alpha)+B\}\,r]\,(1-Br)-(A-B)(1+r)\,r=0,$$

and

$$(1-r^2)\,(A-B)(1-\alpha)+\{(A-B)(1-\alpha)+2B\}(1-r^2)+2\,[\,(L_1K_1)^{1/2}-\{1-((A-B)(1-\alpha)+B)\,Br^2\}]=0\,.$$

The resutl is sharp.

Theorem 3.5. If $f(z)\in D(A,B,\alpha)$, $Re\,g'(z)>0$ in U, then

$$Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}\ \ge\ \begin{cases} M_5(r) \text{ if } R_1\le R_2 \\ M_6(r) \text{ if } R_2\le R_1 \end{cases}$$

where

$$M_5(r)=\frac{1-2r+r^2}{1-r^2}-\frac{(A-B)(1-\alpha)\,r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)},$$

and

$$M_6(r)=\frac{1-2r+r^2}{1-r^2}+\frac{(A-B)(1-\alpha)+2B}{(A-B)(1-\alpha)}+\frac{2[(L_1K_1)^{1/2}-\{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $Re\,g'(z)>0$ in U, we have by Lemma 2.3

$$Re\left\{1+\frac{zg''(z)}{g'(z)}\right\}\ge\frac{1-2r+r^2}{1-r^2}\ .$$

Using this estimate in (3.4) we get the required result on the lines of proof of Theorem 3.2.

To see the bounds are sharp we consider the following two cases:

(i) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,z}{1+B\,z}$$

with $g(z)=-z-2\log(1-z)$, when $R_1\le R_2$

and

(ii) $$\frac{f'(z)}{g'(z)}=\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}$$

and Tuan [1] by putting $\alpha = 0$ and $\beta = 1$.

Lemma 2.2. If $h(z) = 1+c_1z+ \dots$ is regular in U and $Re\ h(z) > 0$ for $z \in U$, then

$$Re\ h(z) \geq \frac{1-r}{1+r},\ |z| = r.$$

This is a well known result due to Caratheodory.

Lemma 2.3. If $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is regular $Re\ g'(z) > 0$ in U, then

$$Re\left\{1 + \frac{z\,g''(z)}{g'(z)}\right\} \geq \frac{1-2r+r^2}{1-r^2},\ |z| = r.$$

The result is sharp.

The above result is contained in the proof of Theorem 4 of Ratti [4].

3. In this section we obtain main results.

Theorem 3.1. If $f(z) \in D(A,B,\alpha)$, then

$$|a_n - b_n| \leq \frac{(A-B)(1-\alpha)}{n},\ n = 2, 3, \dots \text{ and } 0 \leq \alpha < 1.$$

If $Re\,(a_k \overline{b_k}) \leq 0,\ k = 2, 3, \dots$.

The result is sharp.

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\,w(z)}{1+B\,w(z)},$$

or

$$f'(z) - g'(z) = [(A-B)(1-\alpha)\,g'(z) + B\,\{g'(z) - f'(z)\}]\,w(z).$$

That is

$$(3.1)\quad \sum_{n=2}^{\infty} n\,(a_n - b_n)\,z^{n-1} = \{(A-B)(1-\alpha)(1+\sum_{n=2}^{\infty} n\,b_n z^{n-1}) + B\sum_{n=2}^{\infty} n\,(a_n - b_n)\,z^{n-1}\}\sum_{n=1}^{\infty} c_n z^n$$

For $n \geq 2$, we have from (3.1)

$$\sum_{k=2}^{n} k\,(a_k - b_k)\,z^{k-1} + \sum_{k=n+2}^{\infty} d_k z^{k-1} = [(A-B)(1-\alpha) + (A-B)(1-\alpha)\sum_{k=2}^{n-1} k\, b_k z^{k-1} + B\sum_{k=2}^{n-1} k\,(b_k - a_k)\,z^{k-1}]\,w(z)$$

which gives

$$\left|\sum_{k=2}^{n} k\,(a_k - b_k)\,z^{k-1} + \sum_{k=n+1}^{\infty} d_k z^{k-1}\right|^2 \leq \left|(A-B)(1-\alpha) + (A-B)(1-\alpha)\sum_{k=2}^{n-1} k\, b_k z^{k-1} + B\sum_{k=2}^{n-1} k\,(b_k - a_k)\,z^{k-1}\right|^2.$$

On integrating over $|z| = r$, $0 < r < 1$, we get

$$\sum_{k=2}^{n} k\,(a_k - b_k)\, z^{k-1} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k-2} \le (A-B)^2 (1-\alpha)^2 + (A-B)^2 (1-\alpha)^2$$
$$+ (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n-1} k^2 \,|b_k|^2 r^{2k-2} + B^2 \sum_{k=2}^{n-1} k^2 \,|b_k - a_k|^2 r^{2k-2}.$$

If we take the limit as $r \to 1$, then

$$\sum_{k=2}^{n} k^2 \,|a_k - b_k|^2 \le (A-B)^2(1-\alpha)^2 \sum_{k=2}^{n-1} k^2 \,|b_k|^2 + B^2 \sum_{k=2}^{n-1} k^2 \,|b_k - a_k|^2$$

or

$$n^2 |a_n - b_n|^2 \le (A-B)^2(1-\alpha)^2 + (A-B)^2 (1-\alpha)^2 \sum_{k=2}^{n} k^2 |b_k|^2 + B^2 \sum_{k=2}^{n-1}$$
$$k^2 \left[\, |b_k|^2 + |a_k|^2 - 2|a_k|\,|b_k| \,\right] - \sum_{k=2}^{n-1} k^2 \left[\, |b_k|^2 + |a_k|^2 - 2|a_k|\,|b_k| \,\right]$$
$$= (A-B)^2 (1-\alpha)^2 + \sum_{k=2}^{n-1} k^2 \left[\{ (A-B)^2 (1-\alpha)^2 + B^2 - 1\} \,|b_k|^2 + (B^2-1)\,|k|_2 \right.$$
$$\left. + 2 \sum_{k=2}^{n-1} k^2(1-B^2)\, Re\,(a_k\, \overline{b_k}) \right] \le (A-B)^2 (1-\alpha)^2,$$

$$\{A-1 \le B < A \le 1,\ 0 \le \alpha < 1,\ Re\,(a_k\, \overline{b_k}) \le 0\}$$

that is,

$$|a_n - b_n| \le \frac{(A-B)\,(1-\alpha)}{n},\ n = 2, 3, \dots .$$

To establish the sharpness of the result we consider the following funcitons

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}},$$

and $$g(z) = \int_0^z \frac{(1+B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}]^2}.$$

It is obvious that $f(z) \in D(A,B,\alpha)$.

Now

$$f(z) = \int_0^z \frac{dt}{1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}},$$
$$= \int_0^z [1 + \{ (A-B)(1-\alpha) + B\} t^{n-1}]^{-1}\, dt$$
$$= z - \frac{\{ (A-B)(1-\alpha) + B\}}{n} z^n + \dots$$

and

$$g(z) = \int_0^z \frac{(1+B\, t^{n-1})\, dt}{[1 + \{(A-B)(1-\alpha)+B\}\, t^{n-1}]^2},$$
$$= \int_0^z [(1+B\, t^{n-1})\, [1 + \{ (A-B)(1-\alpha) + B\} t^{n-1}]^{-2}\, dt$$

$$= z - \frac{[2\{(A-B)(1-\alpha)+B\} - B]}{n} z^n + \ldots$$

Clearly, $a_n = -\dfrac{\{(A-B)(1-\alpha)+B\}}{n}$ and $b_n = -\dfrac{[2\{(A-B)(1-\alpha)+B\} - B]}{n}$.

Hence

$$|a_n - b_n| = \frac{(A-B)(1-\alpha)}{n}.$$

This shows the sharpness of the result.

Remark. On putting $A = \delta$ and $B = \delta$ and $\alpha = 0$ in Theorem 3.1 the result of Lakshminarasimhan([3], Theorem 5) follows.

Theorem 3.2. If $f(z) \in D(A,B,\alpha)$, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \le \begin{cases} M_1(r) & \text{for } R_1 \le R_2 \\ M_2(r) & \text{for } R_2 \le R_1 \end{cases}$$

where

$$M_1(r) = \frac{1-4r+r^2}{1-r^2} - \frac{(A-B)(1-\alpha)\,r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)}.$$

$$M_2(r) = \frac{1-4r+r^2}{1-r^2} + \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)}.$$

$$+ \frac{2\,[\,(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}\,Br^2\,\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

Proof. Since $f(z) \in D(A,B,\alpha)$, we have

$$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)\,(1-\alpha) + B\}\,w(z)}{1+B\,w(z)},$$

(3.2) Let $\dfrac{f'(z)}{g'(z)} = p(z)$, where $p(z) = \dfrac{1 + \{(A-B)\,(1-\alpha) + B\}\,w(z)}{1+B\,w(z)}$.

On differentiating (3.2) logarithmically, we get

$$\frac{z f''(z)}{f'(z)} - \frac{z g''(z)}{g'(z)} = \frac{z p'(z)}{p(z)}.$$

$$\text{or } Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\frac{z g''(z)}{g'(z)}\right\} = Re\left(\frac{z p'(z)}{p(z)}\right).$$

Using Lemma 2.1 in (3.3), we have

$$\text{(3.4)} \qquad Re\left\{1+\frac{z f''(z)}{f'(z)}\right\} - Re\left\{1+\frac{z g''(z)}{g'(z)}\right\}$$

$$\ge \begin{cases} -\dfrac{(A-B)\,(1-\alpha)\,r}{[1-\{(A-B)\,(1-\alpha)+B\}r]\,(1-Br)}, & \text{if } R_1 \le R_2 \\[2ex] \dfrac{(A-B)\,(1-\alpha)+2B}{(A-B)\,(1-\alpha)} + \dfrac{2[(L_1K_1)^{1/2} - [1-\{(A-B)(1-\alpha)+B\}Br^2]}{(A-B)\,(1-\alpha)\,(1-r^2)} & \text{if } R_2 \le R_1 \end{cases}$$

Since $g(z)$ is univalent, we have [2]

(3.5) $$Re\left\{1+\frac{z\,g''(z)}{g'(z)}\right\} \geq \frac{1-4r+r^2}{1-r^2}, \quad |z| = r.$$

Combining (3.4) and (3.5) we get the required result.
Sharpness of the bounds follows if we choose

(i) $$\frac{f'(z)}{g'(z)} = \frac{1+\{(A-B)(1-\alpha)+B\}z}{1+Bz} \text{ with } g(z) = \frac{z}{(1-z)^2}, \text{ when } R_1 \leq R_2$$

and

(ii) $$\frac{f'(z)}{g'(z)} = \frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)} \quad \text{with } g(z) = \frac{z}{(1-z)^2},$$

when $R_2 \leq R_1$

where $w_1(z) = \dfrac{z\,(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1+\{(A-B)(1-\alpha)+B\}\,w_1(z)}{1+B\,w_1(z)}\right] = R_1 \qquad \text{at } z = -r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 1) turn out to be special case by taking $A = 1$, $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, then $f(z)$ is convex in

$$|z| < \begin{cases} r_1 & \text{for } R_1 \leq R_2 \\ r_2 & \text{for } R_2 \leq R_1 \end{cases}$$

where r_1 and r_2 are respectively the smallest positive roots of the following equations :

(3.6) $$(1-4r+r^2)\,[1-\{(A-B)(1-\alpha)+B\}\,r]\,(1-Br) - (A-B)(1-\alpha)\,(1-r^2)\,r = 0$$

(3.7) $$(1-4r+r^2)\,\{(A-B)(1-\alpha)\} + \{(A-B)(1-\alpha)+2B\}(1-r^2)$$
$$+2\,[\,(L_1K_1)^{1/2} - \{1-((A-B)(1-\alpha)+B)\,Br^2\}] = 0.$$

The result is sharp.

Theorem 3.3. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then

$$Re\left\{1+\frac{z\,f''(z)}{f'(z)}\right\} \geq \begin{cases} M_1(r) & \text{if } R_1 \leq R_2 \\ M_2(r) & \text{if } R_2 \leq R_1 \end{cases}$$

where $M_1(r)$ and $M_2(r)$ are same as in Theorem 3.2.

The result is sharp.

Proof. Since $g(z)$ is starlike in D implies $g(z)$ is univalent there, the proof of this theorem follows on the steps Theorem 3.2.

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 2) turn out to be special case by taking $A = 1$ and $B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and starlike in U, then $f(z)$ is convex in

$$|z| < \begin{cases} r_1 & \text{for } R_1 \le R_2 \\ r_2 & \text{for } R_2 \le R_1 \end{cases}$$

where r_1 and r_2 are respectively the smallest positve roots of the equations (3.6) and (3.7).

Theorem 3.4. If $f(z) \in D(A,B,\alpha)$, $g(z)$ is regular and convex in U, then

$$Re\left\{1 + \frac{z f''(z)}{f'(z)}\right\} \ge \begin{cases} M_3(r) \text{ if } R_1 \le R_2 \\ M_4(r) \text{ if } R_2 \le R_1 \end{cases}$$

where

$$M_3(r) = \frac{1-r}{1+r} - \frac{(A-B)(1-\alpha)\, r}{[1-\{(A-B)(1-\alpha)+B\}r]\,(1-Br)},$$

and

$$M_4(r) = \frac{1-r}{1+r} - \frac{\{(A-B)(1-\alpha)+2B\}}{(A-B)(1-\alpha)} + \frac{2[(L_1K_1)^{1/2} - \{1-(A-B)(1-\alpha)+B\}Br^2\}]}{(A-B)(1-\alpha)\,(1-r^2)}$$

The result is sharp.

Proof. Since $g(z)$ is convex in U, therefore $g'(z) \neq 0$ in U and

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} > 0 \text{ in } U.$$

The function

$$1 + \frac{z g''(z)}{g'(z)} = 1 + c_1 z + \ldots$$

is regular in U and has positive real part, therefore by Lemma 2.2

$$Re\left\{1 + \frac{z g''(z)}{g'(z)}\right\} \ge \frac{1-r}{1+r}, \quad |z| = r.$$

Using this estimate in Theorem 3.4, we get the required result on the lines of proof of Theorem 3.2.

To see that the bounds are sharp, we consider the following two cases.

(i) $$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, z}{1+Bz} \text{ with } g(z) = \frac{z}{1-z}, \text{ when } R_1 \le R_2$$

and

(ii) $$\frac{f'(z)}{g'(z)} = \frac{1 + \{(A-B)(1-\alpha) + B\}\, w_1(z)}{1+B\, w_1(z)} \text{ with } g(z) = \frac{z}{1-z}, \text{ when } R_2 \le R_1$$

where $w_1(z) = \dfrac{z(z-c)}{1-cz}$ with c defined by the condition

where $w_1(z) = \dfrac{z\,(z-c)}{1-cz}$ with c defined by the condition

$$Re\left[\frac{1 + \{(A-B)\,(1-\alpha) + B\}\, w_1(z)}{1+B\, w_1(z)}\right] = R_1 \qquad \text{at } z = -r.$$

Remark. From the above theorem the following result can be obtained immediately from which result of Ratti ([4], Theorem 4) turn out to be special case by taking $A = 1,\ B = 0$ and $\alpha = 0$.

Corollary. If $f(z) \in D(A,B,\alpha)$, $Re\, g'(z) > 0$ in U, then $f(z)$ is convex in

$$|z| < \left[\begin{array}{ll} r_5 & \text{for } R_1 \le R_2 \\ r_6 & \text{for } R_2 \le R_1 \end{array}\right.$$

where r_5 and r_6 respectively the smallest positive roots of the following equations:

$$(1-2r-r^2)\,[1-\{(A-B)\,(1-\alpha)+B\}\,r]\,(1-Br) - (A-B)(1-\alpha)\,(1-r^2)\,r = 0$$

and

$$(1-2r-r^2)\,\{(A-B)\,(1-\alpha)\} + \{(A-B)(1-\alpha) + 2B\}(1-r^2) + 2\,[\,(L_1K_1)^{1/2} - \{1-((A-B)\,(1-\alpha)+B)\,Br^2\}] = 0.$$

The result is sharp.

REFERENCES

[1] V.V. Anh and P.D. Tuan, On β-Convexity of certain starlike univalent functions, *Rev. Roum. Math. Pures Et Appl. Math.* **5** (1974), 701-707.

[2] W.K. Hayman, *Multivalent Functions*, Cambridge University Press, London, 1967.

[3] T.V. Lakshminarasimhan, On subclass of functions starlike in the unit disc, *J. Indian Math. Soc.* **41** (1977), 233-243.

[4] J.S. Ratti, The radius of convexity of certain analytic functions II, *Inter. J. Math. and Math. Sci.* **3** (1980), 483-489.

Jñānābha, Vol. 30, 2000

ON FLEXIBILITY OF PRIME ASSOSYMMETRIC RINGS

By

K. Suvarna and G. Ramabhupal Reddy

Department of Mathematics, Sri Krishnadevaraya University, Anantapur-515 003, Andhra Pradesh, India

(Received : April 25, 1999)

ABSTRACT

In this paper we show that a non-associative 2-and 3-divisible prime assosymmetric ring is flexible.

1. Introduction. E. Kleinfeld [1] introduced a class of non-associative rings called as assosymmetric rings in which the associator $(x,y,z) = (xy)z - x(yz)$ has the property $(x,y,z) = (p(x), p(y), p(z))$ for each permutation p of x,y and z. These rings are neither flexible nor power-associative. In [1] it is proved that the commutator and the associator are in the nucleus of this ring. By using these properties we show that a non-associative 2-and-3-divisible prime assosymmetric ring is flexible.

2. Preliminaries. Throughout this papr R will denote a non-associative 2-and 3-divisible assosymmetric ring. The commutator (x,y) of two elements x and y in a ring is defined by $(x,y) = xy - yx$. The nucleus N in R is the set of elements $n \in R$ such that $(n,x,y) = (x,n,y) = (x,y,n) = 0$ for all x,y in R. The center C of R is the set of elements $c \in N$ such that $(c,x) = 0$ for all x in R. A non-associative ring R is called flexible if $(x,y,x) = 0$ for all x,y in R. A ring is said to be power–associative if every subring of it generated by a single element is associative Let I be the associator ideal of R. I consists of the smallest ideal which contains all associators. R is called k-divisible if $kx = 0$ implies $x = 0$, $x \in R$ and k is a natural number.

In an arbitrary ring the following identities hold:

(1) $(wx,y,z) - (w, xy, z) + (w,x,yz) = w(x,y,z) + (w,x,y)z$

$f(w,x,y,z) = (wx,y,z) - x(w,y,z) - (x,y,z)w$

and

(2) $(xy,z) - x(y,z) - (x,z)y = (x,y,z) - (x,z,y) + (z,x,y)$.

In any assosymetric ring (2) becomes

(3) $(xy,z) - x(y,z) - (x,z)y = (x,y,z)$

It is proved in [1] that in a 2-and 3-divisible assosymetric ring R the following identities hold for all w,x,y,z,t in R.

(4) $f(w,x,y,z) = 0$, that is, $(wx,y,z) = x(w,y,z) + (x,y,z)w$,

(5) $((w,x), y,z) = 0$

and

(6) $((w,x,y), z,t) = 0$.

That is, every commutator and associator is in the nucleus N. From

(3), (5) and (6), we obtain

(7) $x(y,z) + (x,z)y \subset N$.

Suppose that $n \in N$. Then with $w = n$ in (1) we get $(nx,yz) = n(x,y,z)$. Combining this with (5) yields.

(8) $(nx,y,z) = n(x,y,z) = (xn,y,z)$

From (7) and (8), we obtain

(9) $(y,z)(x,r,s) = -(x,z)(y,r,s)$.

3. Main Results

Lemma 1. Let $S = \{s \in N \mid s(R,R,R) = 0\}$. Then S is an ideal of R and $S(R,R,R) = 0$

Proof. By subsituting s for n in (8), we have $(sx,y,z) = s(x,y,z) = (xs,y,z) = 0$. Thus $sR \subset N$ *and* $Rs \subset N$. From (6), $sw(x,y,z) = s.\ w(x,yz)$. But (1) multiplied on the left by s yields $s.w(x,yz) = -s(w,x,y)z = -s(w,x,y).\ z = 0$. Thus $sw.\ (x,y,z) = 0$. From (9), we have $(s,w)(x,y,z) = -(x,w)(s,y,z) = 0$. Combining this with $sw.(x,y,z) = 0$, we obtain $ws.(x,y,z) = 0$. Thus S is an ideal of R. The rest is obvious. This completes the proof of the lemma.

Lemma 2. $(x,y,x) \in S$.

Proof. By forming the associatiors of both sides of (1) with u and v, and using (6), we obtain

(10) $(w(x,y,z), u,v) + ((w,x,y)z,u,v) = 0$.

Interchanging y and z in (10) and subtracting the result from (10), we get

(11) $((w,x,y)z,u,v) = ((w,x,z)y,u,v)$.

But $((w,x,z)y,u,v) = (y(w,x,z),u,v)$, because of (5). So that

(12) $((w,x,y)z,u,v) = (y(w,x,z),u,v)$, as a result of (11).

Also by permuting w and y in (10), we obtain $(y(w,x,z),u,v) + ((w,x,y)z,u,v) = 0$. This identity with (12) yields $2((w,x,y)z,u,v) = 0$. Thus

(13) $((w,x,y)z,u,v) = 0$.

From (6) we have $(x,y,x) \subset N$. Using (13) and (8), we get $0 = ((x,y,x)z,u,v) = (x,y,x)(z,u,v)$ for all x,y,z,u,v in R. Hence $(x,y,x) \in S$. This completes the proof of the lemma

Theorem. If R is a non-associative 2-and 3-divisible prime assosymmetric ring, then R is flexible.

Proof. Using lemma 1 and (1) we establish that $S.\ I = 0$. Since R is prime, either $S = 0$ or $I = 0$. If $I = 0$, R is associative. But we have assumed that R is not associative. Therefore $I \neq 0$. Hence $S = 0$. From lemma 2, $(x,y,x) \in S$. Thus $(x,y,x) = 0$. That is, R is flexible.

REFERENCES

[1] E. Kleinfeld, Assosymmetric rings. *Proc. Amer. Math. Soc.* 8 (1957), 983-986.
[2] E. Kleinfeld, Rings with (x,y,x) and commutators in the left nucleus. *Comm. in Algebra,* 16 (10) (1988), 2023-2029.

Jñānābha, Vol. 30, 2000

NUMERICAL SOLUTION OF THE STEADY STATE NAVIER-STOKES EQUATIONS USING ELLIPTIC SOLVER

By

P.K. Sharma
Department of Mathematics, D.A.V. Postgraduate College, DehraDun-248001, Uttaranchal, India
and
Rajesh Kumar
Geodetic and Research Branch, Survey of India
DehraDun-248001, Uttaranchal, India

(Received : April 30, 1999)

ABSTRACT

The motion of a liquid under action of viscous forces is governed by Navier-Stokes equations. Exact solutions of *N-S* equations are available only under some ideal assumptions. Tremendous advances in the development and use of numerical methods, mainly due to availability of fast computing devices, have given rise to various types of numerical solutions of these equations. Driven cavity problem has worked as an ideal prototype, therefore present numerical scheme, which uses a fast direct elliptic equation solver is also tried on this problem. The results obtained compare well with those of previous authors.

1. Introduction. The Navier-Stokes equations for two dimensional steady flow of an incompressible fluid may be written in the vorticity *(w)* and streem function *(ψ)* formulation as :

$$\nabla^2\omega - Re\,(\psi_y\,\omega_x - \psi_x\omega_y) = 0 \qquad \text{... (1)}$$

$$\nabla^2\psi = -\,\omega \qquad \text{... (2)}$$

satisfying

$$u = \frac{\partial\psi}{\partial y},\; v = -\frac{\partial\psi}{\partial x} \qquad \text{... (3)}$$

and

$$w = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \text{... (4)}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$$

The equation (1) and (2) can be formulated as follows.

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} - Re\left\{\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right\} = 0 \quad \text{... (5)}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -w \quad \text{... (6)}$$

The equations (5) - (6) together with boudary conditions constitutes a non-linear elliptic boundary value problem. The degree of non-linearity increases with Reynolds number.

The numerical solution procedure generally consists of discretizing the domain over which equations (5) -(6) are defined. This gives two systems of algebraic equations. One can obtain the solution in terms of stream function and vorticity.

Burggraf [3] was first to employ central difference scheme for the solution of equations (5)-(6). The resulting systems of eqations were solved by using iterative procedures. But Iterative procedures fail to converge even at moderate Reynolds number. This difficulty originated the idea of upwind schemes. However, it is well known that upwind differncing introduces false diffusion effects and this will lead to additional error into the numerical solution as shown by Strikwerda [21]

Roache [16] gave the idea of what he calls Laplacian Driver Method. It is worth emphasizing that the superiority of Laplacian Driver Method over other methods is its simplicity to apply on computer, because one has to solve two Poisson's equations iteratively. We have applied an elliptic equation solver to solve equations (5) and Poisson's equation solver for equation (6). The system of equations (5) and (6) is solved iteratively until a desired convergence-criterion is satisfied.

Roache [16] has reported the solution of the driven cavity flow problem for Reynolds number $Re = 20$ with a 11×11 mesh. By using our technique, we have obtained the solutions upto Reynolds number $Re =$ 500 with a 21×21 mesh. The results obtained by us compare well with those of previous authors.

2. Derivation of Boundary Conditions. Since the values of vorticity on the boundary are not explicitly given, We compute these with the help of derivative boundary conditions on ψ Assuming equation (5) is valid near the wall and using n to represent the coordinate normal to the wall.

$$w(1) = -\left[\frac{\partial^2 w}{\partial n^2}\right]_1 \quad \text{... (7)}$$

where the argument and subscript '1' denote the mesh point on the wall. Using a Taylor's series expansion for ψ with step length Δn about the mesh point on the wall.

$$\psi(2) = \psi(1) + \Delta n\left[\frac{\partial\psi}{\partial n}\right]_1 + \frac{\Delta n^2}{2}\left[\frac{\partial^2\psi}{\partial n^2}\right]_1 + \frac{\Delta n^3}{6}\left[\frac{\partial^3\psi}{\partial n^3}\right]_1 + O(\Delta n^4) \qquad ... (8)$$

$$\psi(3) = \psi(1)+2\Delta n\left[\frac{\partial\psi}{\partial n}\right]_1 + \frac{4\Delta n^2}{2}\left[\frac{\partial^2\psi}{\partial n^2}\right]_1 + \frac{8\Delta n^3}{6}\left[\frac{\partial^3\psi}{\partial n^3}\right]_1 + O(\Delta n^4) \qquad ... (9)$$

From (8) and (9) and making use of (7), we get

$$w(1) = \left[\psi(3) - 8\psi(2) + 7\psi(1) + 6\Delta n\left\{\frac{\partial\psi}{\partial n}\right\}_1\right] / (2\Delta n^2).$$

Since ψ (1) is zero on all four boundaries, the above equation reduces to

$$w(1) = \left[\psi(3) - 8\psi(2) + 6\Delta n\left\{\frac{\partial\psi}{\partial n}\right\}_1\right] / (2\Delta n^2) \qquad ... (10)$$

This is second order approximation for vorticity and was used by Bozeman and Dalton [2]. Roache [16] calls it Jensen's fourmula. It is also referred as Briley's formula. Gupta and Manohar [9] have investigated the effects of boundary approximations on the solution and shown that Briley's formula gives more accurate results than any of the other formula.

3. Methematical Model. From a computational viwepoint, the cavity flow is an ideal prototype non-linear problem which is readily posed fo numerical solution. Because of its geometrical simplicity and comparatively minor singularities, it has served as a model problem for last twenty years for testing new numerical schemes and as a benchmark solution for making comparisons among various schemes. Fortunately, a number of numerical methods have been tried on this problem and results have been published. We have also applied our technique for the solution of this classical problem and obtained results which are comparable with the previous results.

The cavity, as shown in fig. (A) is rectangular in cross-section and filled with a Newtonian, viscous and incompressible fluid. The fluid is forced to make by the motion of the upper surface which travels with constant linear velocity in its own plane. The cavity is assumed to be long in the longitudinal (Z) direction so that the fluid motion is essentially two dimensional.

The formation of the governing equation for steady motion has been described in detail by Mills [13] and Burggraf [3]. The steady flow in the square cavity (fig. (B)) is governed by the equations (5) and (6) together with the under mentioned boundary conditions.

The boundary conditions for ψ may be derived from the conditions that each wall is impermeable and that the viscous boundary conditions

implies that the fluid adjacent to the wall moves with the same velocity as the wall. The first condition implies that $\psi = 0$ on each boundary and second that the derivative of ψ in direction normal to the wall is equal to the tangential velocity of the wall. The boundary conditions thus obtained for driven cavity problem are as under :

$\psi = 0$ along all boundaries ... (11)

$\psi_x = 0$ along vertical walls OC and AB ... (12)

$\psi_y = 0$ along the bottom wall *OA* ... (13)

$\psi_y = -1$ along th sliding wall *BC*. ... (14)

4. Different Approaches Tried for This Problem. Burggraf [3] was first to employ central difference schemes for this problem using iterative procedures. Since the diagonal dominance property is not nacessarily satisfied, the standard iterative procedures such as Gauss-Seidel or SOR fail to converge rapidly even at moderate high Reynolds number. Later Spalding [20] discovered that stable solution could be obtained if upwind differencing methods were used for the convective terms in the vorticity equation. Further, Finite difference calculations using similar ideas have been presented by Nallaswamy and Krishna Prasad [14]. However, recently Strikwerda [21] and others have brought out clearly the drawbacks of upwind schemes.

The second approach to overcome the convergence problem is that at each outer iteration, the algebraic equations corresponding (5) - (6) are solved by direct fast solvers. Gupta [8] has applied this approach and used direct solver *MA 28* from the Harwell Package [5]. But These solvers do not take the benefit of sparsity into consideration and thus are not economical in computer-time and storage.

5. Difference Equations. The rectangular region over which the equations (1) and (2) are defined is divided into a uniform mesh by choosing mesh widths h along x-direction and k along y-direction. Applying central difference formulae to (1) and (2), relative to a uniform mesh system, we obtain

$$A_1 w_{i,j-1} + A_2 w_{i-1,j} + A_3 w_{i,j} + A_4 w_{i+1,j} + A_5 w_{i,j+1} = A_6 \quad \text{... (15)}$$

where

$$A_1 = \frac{1}{k^2} + \frac{Re}{4hk}(w_{i+1,j} - w_{i-1,j})$$

$$A_2 = \frac{1}{h^2} + \frac{Re}{4hk}(w_{i,j+1} - w_{i,j-1})$$

$$A_3 = -2\left[\frac{1}{h^2} + \frac{1}{k^2}\right]$$

$$A_4 = \frac{1}{h^2} + \frac{Re}{4hk}\,(w_{i\,,\,j+1} - w_{i\,,\,j-1})$$

$$A_5 = \frac{1}{k^2} + \frac{Re}{4hk}\,(w_{i+1\,,\,j} - w_{i-1\,,\,j})$$

$$A_6 = 0$$

and

$$B_1\psi_{i,\,j-1} + B_2\,\psi_{i-1\,,\,j} + B_3\psi_{i\,,\,j} + B_4\,\psi_{i+1\,,\,j} + B_5\psi_{i\,,\,j+1} = B_6 \qquad \ldots (16)$$

where

$$B_1 = \frac{1}{k^2},\;\; B_2 = \frac{1}{h^2},\;\; B_3 = -2\left[\frac{1}{h^2} + \frac{1}{k^2}\right],\; B_4 = \frac{1}{h^2},\;\; B_5 = \frac{1}{k^2},\; B_6 = -w_{i\,j}.$$

6. Computational Procedure. Following are the steps involved in the computational procedure.

(a) Assign initial approximation $\psi^{(m)}$ with $m = 0$. Here approximation is taken as $\psi^{(0)} = 0$.

(b) Compute boundary values for ω by boundary conditions as in (10).

(c) Solve for Vorticity $\varpi^{(m+1)}$ from

$$A_1 w_{i,\,j-1}^{m+1} + A_2\, w_{i-1,\,j}^{m+1} + A_3\, w_{i,\,j}^{m+1} + A_4\, w_{i+1,\,j}^{m+1} + A_5 w_{i,\,j+1}^{m+1} = A_6$$

(d) Damp values of Vorticity in interior of domain as flollows

$$\omega^{(m+1)} = \delta\varpi^{(m+1)} + (1-\delta)\,\omega^{(m)}, \qquad 0 < \delta < 1.$$

(e) Solve stream function equation to obtain $\psi^{(m+1)}$ as

$$B_1\psi_{i,\,j-1}^{m+1} + B_2\,\psi_{i-1,\,j}^{m+1} + B_3\,\psi_{ij}^{m+1} + B_4\,\psi_{i+1,\,j}^{m+1} + B_5\psi_{i,\,j+1}^{m+1} = B_6 \qquad \ldots (16)$$

(f) Damp values of stream function as

$$\psi^{(m+1)} = \delta\psi^{(m+1)} + (1-\delta)\,\psi^{(m)}$$

(g) Repeat steps (b) - (f) until following convergence criterion is satisfied :

$$max\ |\ \omega_{i,\,j}^{(m+1)} - \omega_{i,\,j}^{(m+1)}\ | < \epsilon\,.$$

(h) After above convergence criterion is satisfied, boundary values for ω are again computed as in (b) and solution is attained over whole domain.

7. Algorithm. The resulting structure of system of equation (15)-(16) can be rewritten as :

$$A\overline{U} = \overline{V} \qquad (17)$$

Coefficient matrix A is a block tri-diagonal system having n blocks where each block is of order (mxm). $\overline{U}$ is a column matrix of unknown and $\overline{V}$ is also a column matix containing right hand sides of equations. Since diagonal dominance property in such a system does not always hold, so

iterative methods sometimes fail to converge. Recently some efficient direct methods have been employed to solve such a system. Linger [12] has applied a sem-direct method to solve Poisson's equation.

Elsner and Mehramann [6] have dealt in detail on the convergence condition of block iterative methods. The block tri-digonal system of linear equations is a sparse coefficient matrix system and it is possible to take the advantage of the sparsensess in order to reduce both computation time and storage requirements. Duff [4], Erisman and Reid [7] have dealt in detail with the direct methods for sparse matrices. Jennings [11] has also given methods like elimination using submatrices to deal with a sparse structure of such type.

Present algorithm also employs a direct elimination technique to solve block tri-diagonal system of linear equations. Imortant steps in brief are as follows. Detailed analysis of algorithm and storage etc. are given in Sharma and Agarwal [19].

The system (13) is written in matrix form as follows.

$$\begin{bmatrix} A_1 & B_1 & & & \\ C_2 & A_2 & B_2 & & \\ & \ddots & \ddots & \ddots & \\ & & C_{n-1} & A_{n-1} & B_{n-1} \\ & & & C_n & A_n \end{bmatrix}_{mn\times mn} \begin{bmatrix} \overline{U}_1 \\ \vdots \\ \overline{U}_{n-1} \\ \overline{U}_n \end{bmatrix} = \begin{bmatrix} V_1 \\ \vdots \\ V_{n-1} \\ V_n \end{bmatrix}_{mn\times 1} \quad \text{... (18)}$$

where A_j' s, $j = 1\ (1)\ n$ are tri-diagonal matrices, B_j's and C_j's, are diagonal matrices given by:

$$\begin{bmatrix} \beta_{1j} & \gamma_{1j} & & & \\ \alpha_{2j} & \beta_{2j} & \gamma_{2j} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \beta_{m-1,j} & \gamma_{m-1,j} \\ & & & \alpha_{mj} & \beta_{mj} \end{bmatrix}_{m\times m} \quad \text{... (19)}$$

$$B_j = Diag\ [\delta_{1j}, \delta_{2j}, \dots \delta_{mj}]_{m\times m} \quad \text{... (20)}$$

$$C_j = Diag\ [\eta_{1j}, \eta_{2j}, \dots \eta_{mj}]_{m\times m} \quad \text{... (21)}$$

where $j = 1, 2, \dots, n$

$\overline{u}_j$, $j = 1\ (1)n$ is unkown column vector defined as

$\overline{u} = [u_{1j}, u_{2j}, \dots, u_{mj}]^T$

V_j, $j = 1\ (1)\ n$ is right hand side modified after necessary adjustment due to given boundary conditions.

$V = [V_{1j}, V_{2j}, \ldots V_{mj}]^T$.

Let R_{ij}, $i = 1(1)m$, $j = 1(1)n$ denotes i^{th} row of j^{th} block. Elimination algorithm is as follows :

Step 1.

For $j = 1$

Do $R_{i,1} = R_{i,1} - (\alpha_{i,1}/\beta_{i-1,1})^*(R_{i-1,1})$ for $i = 2$ to m.

Step 2.

For $j = 1$

(a) $i = 1$

(b) (i) $k = 1$

$R_{i,j} = R_{i,j} - (\eta_{\,i,j}/\beta_{k,\,j-1})^*(R_{k,\,j-1})$.

By this operation, first entry of first row of second block becomes zero and second entry of first row of second block becomes non-zero, say $\eta_{\,i,j}$.

(ii) For $k = k+1$ up m do

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

Where "–" refers to modified entries.

(c) $i = i + 1$

(i) $R_{i,j} = R_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

(ii) for $k = i$ to m (just like 2(b))

$\overline{R}_{i,j} = \overline{R}_{i,j} - (\eta_{\,i,j}/\overline{\beta}_{k,\,j-1})^*(\overline{R}_{k,\,j-1})$

(iii) $\overline{R}_{i,j} = \overline{R}_{i,j} - (\alpha_{i,j}/\overline{\beta}_{i-1,j})^*(\overline{R}_{i-1,j})$.

(d) Again go to step 2(c) upto $i = m$

Step 3.

(i) Put $j = j+1$ and repeat steps 2(a) to 2 (d)

(ii) Continue this process upto $j = n$.

Step 4.

Now the system is reduced to upper triangular form and by back substitution process, we may obtain values of unknowns u_{ij}, $i = 1(1)m$, $j=1(1)n$.

8. Result and Discussions. Computations are made taking uniform mesh-size $h = 1/20$ for various Reyonlds number. The point at which the value of ψ attains it's absolute maximum is called the center of primary vortex (vc). We denote the values of ψ and at the vortex center by ψ_{max} and ω_{vc} respectively. We also give the value of drag-coefficient on the sliding wall defined by-

$$cd = \frac{2}{Re}\int_0^1 \omega\ (s,1)\ ds = \frac{2}{Re}\,\overline{F}$$

where $\overline{F}$ is the value of shear force on sliding wall. The integral here is obtained by using Simpson's one-third rule over mesh points on the sliding wall.

It is evident from the table (I), (II) and (III) that resuts obtained compare well with those of previos authors [[2],[9], [14], [8], [10], [17], [18], [1]]. Streamlines and eqivorticities curves for different Reynolds number have been analysed. It is clear that there is no secondary vortex at $Re = 1$ and 10, but there exists two secondary vorticities at the downstream corners for Re =100 to Re = 500. Also the size of secondary vortices increases with the increase in Reynolds number as observed experimentally by Pan and Acrivos [15]. The equivorticity curves become more asymmetrical and recirculating eddies become more dominant with the increase in Reynolds number. The equivorticity curve at Re = 500 has a secondary eddy on the bottom wall at the level -1.0. The same nature of vorticity curve was observed by Ghia, Ghia and Shin [10] at Re = 400 using a much finer mesh.

TABLE -1

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for Re =1 and 10.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference	
1	0.0978	3.3553	(0.35,0.75)	21.1438	ψ_{max} = 0.0982	h=1/20	(9)
					= 0.0995	h=1/20	(9)
					= 0.0995	h=1/20	(8)
					= 0.1082	h=1/20	(9)
					ω_{vc} = 2.95	h=1/20	(9)
					= 3.02	h=1/20	(9)
					= 3.02	h=1/20	(8)
					= 3.33	h=1/20	(9)
10	0.1130	3.3496	(0.35,0.75)	2.1170	ψ_{max} = 0.1043	h=1/30	(17)
					= 0.1095	h=1/20	(9)
					ω_{vc} = 3.570	h=1/20	(9)
					= 3.155	h=1/50	(14)

TABLE -II

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re* =*100* and *400*.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference	
100	0.113	3.3496	(0.35,0.75)	0.2321	ψ_{max} = 0.1043	h=1/30	(17)
					= 0.1095	h=1/20	(9)
					ω_{vc} = 3.570	h=1/20	(9)
					= 3.155	h=1/50	(14)
400	0.1027	2.2363	(0.40,0.60)	0.0763	ψ_{max} = 0.1129	h=1/40	(18)
					= 0.1139	h=1/30	(10)
					= 0.0970	h=1/20	(1)
					ω_{vc} = 2.2810	h=1/40	(18)
					= 2.2947	h=1/30	(10)
					= 2.3600	h =1/20	(1)

TABLE -III

Values of Primary Vortex (ψ_{max}), Vorticity at the Vortex center (ω_{vc}) and Drag coefficient C_D for *Re* =500 and 1000.

Re	ψ_{max}	ω_{vc}	(x,y)	C_D	Results	Reference	
500	0.1009	2.0132	(0.45,0.60)	0.0641	ψ_{max} = 0.1024	h=1/20	(8)
					ω_{vc} = 2.0504	h=1/20	(8)
					= 1.9048	h=1/20	(8)
1000	0.07914	1.6399	(0.45,0.60)	0.0367	ψ_{max} = 0.0812	h=1/50	(2)
					= 0.0972	h=1/20	(8)
					ω_{vc} = 1.7452	h=1/20	(8)

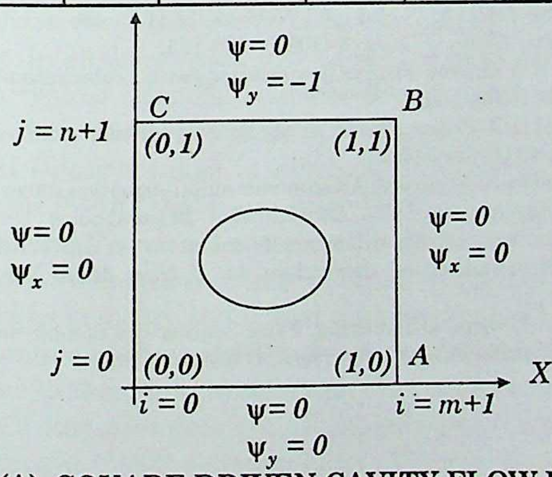


FIG. (A)- SQUARE DRIVEN CAVITY FLOW PROBLEM

REFERENCES

[1] K.E. Barrett and G. Demunshi, Finite Element solutions of convective diffusion problems, *Int. J. Num. Meth. Eng.* **14** (1979), 1511-1524.

[2] J.D. Bozeman and C. Dalton, Numerical study of viscous flow in a cavity, *J. Comp. Phys.*, **12** (1973), 348-363.

[3] O.R. Burggraf, Analytical and numerical studies of the structure of steady separated flows, *J. Fluid Mech.*, **24** (1966), 113-151.

[4] I. Duff, A. Erisman and J. Reid, *Direct Methods for Sparse Matrices*, Oxford Universtiy Press, 1987.

[5] I.S. Duff, *MA28-A* set of *FORTRAN* subroutines for sparse unsymmetric linear equations, *Report AERE-R8730, A.E.R.E. Harwell, U.K.* (1977).

[6] L. Elsner and V. Mehramann, Convergence of block iterative methods for linear systems arising in the numerical solution of Euler Equations, *Numer. Math.*, **59** (1991) 541-559.

[7] A. Erisman and J. Reid, *Direct Methods for Sparse Matrices*, Oxford Universtiy Press, 1981.

[8] M.M. Gupta, A comparison of numerical solution of convective and divergence forms of the Navier-Stokes equations for the driven cavity flow problem, *J. Comp. Phys.*, **43** (1981) 260-267.

[9] M.M. Gupta and R.P. Manohar, Boundary approximations and accuracy in viscous flow computations, *J. Comp. Phys.*, **31** (1979) 265-287.

[10] U. Ghia, K.N. Ghia and C.T. Shin, High resolution for incompressible flow using the Navier-Stokes equations and a multi grid method, *J. Comp. Phys.*, **48** (1982) 387-411.

[11] A. Jennings, *Matrix Computation for Engineers and Scientists*, John Wiley and Sons, 1977.

[12] W. Linger, A guassi-direct fast Poission solver for general regions, *J. Comput. Appl. Math.* (1989), 25-47.

[13] R.D. Mills, On the closed motion of a fluid in a square cavity, *J. Roy, Aero. Soc.*, **69** (1965) 116-120.

[14] M. Nallaswamy and K. Prasad, On cavity flow at high Reynolds numbers, *J. Fluid Mech.* **79** (1977), 391-414.

[15] F. Pan and A. Acrivos, Steady flows in rectangular cavities, *J. Fluid Mech.* **28** (1967), 643-655.

[16] P.J. Roache, The *LAD, NOS* and split *NOS* methods for the steady state Navier-Stokes equations, *Comp. Fluids*, **3** (1975), 179-195.

[17] S.G. Rubin and R.A. Graves, Viscous flow solutions with a cubic spline approximation, *Comp. Fluids*, **3** (1975), 1-36.

[18] R. Schreiber and H.B. Keller, Driven cavity flows by efficient numerical techniques, *J. Comp. Phys.*, **49** (1983), 310-333.

[19] P.K. Sharma and M.K. Aggarwal, An economic elliptic equation solver for rectangular regions, *J. of Engg. and Appl. Sci., Saudi Arabia*, **10** (to Appear)

[20] D.B. Spalding, A novel finite difference formulation for differential expressions involving both first and second derivatives, *Int. J. Num. Meth. Eng.*, **4** (1972), 551-559.

[21] J.C. Strikwerda, Upwind differencing, False Scaling and non-physical solutions to the driven cavity problem, *J. Comp. Phys.*, **47** (1982) 303-307.

Jñānābha, Vol. 30, 2000

WAITING TIME DISTRIBUTION IN A REPAIR SYSTEM WITH MULTIDIMESIONAL STATE SPACE AND RANDOM BATCH ARRIVAL FOR EXCHANGABLE ITEMS

By

S.S. Mishra and S. Rizvi

Department of Mathematics and Statistics,

Dr. R.M.L. Avadh University, Faizabad-224001, U.P., India

(Received : July 5, 1999)

ABSTRACT

In the present paper, we consider multidimensional state space with random batch arrival and compute the probability distributions of the waiting times under steady state for exchangable items in the system.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov forward Equations.

1. Introduction. Stochastic networks are models for flexible manufacturing system (*FMS*) computer networks, communication systems and complex repair system etc. From a customer's point of view, perhaps the most important performance measure of a such system is the sojourn time (or moment of it) in the network or in parts of it. These sojourn times represent (individual) production or repair times for an items, the time for a data packet to be transmitted by the communication system etc. In all these cases, we observe the individual possibly random route of a typical (tagged) customer from arrival at the system until departure. For a review of the results concerning the distribution of these individual sojourn times in stochastic networks, see for example Schassberger [5] and Diney and keing [4].

Breg and Posner [1] observe that these are important problems with sojourn times (delay times) in repair system or *FMS* that can not be modeled by standard individual sojourn times. These problems arise (in the example of Breg and Posner [1], p. 287) when customers bring failed items to the repair system and the items are exchangable in the sense that a customer does not necessarily want to back the particular item be brought into the system but rather any good item of the same kind. They distinguish a "Customer queue" under a first come first served

(FCFS) discipline where the customers wait for any repaired item and "item queue" of ***M/M/C-FCFS*** system where the items wait for repair at one of the repair channels. The multichannel structure of the repair system implies the possibility of overtaking for items such that customer does not necessarily obtain his own item back.

While delay times usually are computed for individual items at the item queue (which is easy to do for the *M/M/C* system under consideration) Breg and Posner [1] investigated delay times at the customer queue, a problem which turned out to be much more complicated. In addition, they introduce spares at the service centre which, if at hand, may be handed immediately to customers who brought in failed items.

Clearly, such problems arise in general complex repair system or in *FMS* (for an example see Schroder [6]) which leads to the questions : what can be said about the delay distribution when customers are waiting in a queue (under some general discipline) for exchangable items which migrate through a general stochastic network.

A careful investigation of the proof of Berg and Posner [1] shows that it strongly depends on the fact that the state space of the system is one dimensioanl and it turns out that the necessity of more complicated state space to describe general (multiqueue) repair systems which lead to some apparently new problems. They considered the simplest system with two dimensional state space i.e. two different parallel sources each having its own queues under ***FCFS*** and computed the steady state probability distribution of the waiting times for exchangable items in that systems. They preferred two statespace case avoiding complexities of probabilty distribution, their generating functions and *LSTs* in the analysis of the multistate problem.

In the present work, we consider multidimensional state space with random batch arrival and compute the probability distribution of the waiting times under steady state for exchagable items in the system assuming that size of the batch of the customers be represented by random variable X such that $E(X) = a$.

We finally discuss the working and non-working states of the system and present the transitional probability distributions of it under these two states by making use of Markov Process and Kolmogorov Forward Equations.

2. The System. Customers arrive to the system (see figure) in Poisson stream of intensity a $\lambda > 0$ at a repair station deliver a failed item to the station and proceed to customer queue, which is organized by using ***FCFS*** rule. The items ae assumed to be exchangable that is

customers accept any repaired item that is given to them. An item leaving the repair station is given to the customer at the head of queue which immediately departs from the queue.

The repair station consists of n-batch servers each having its own queue with queueing discipline ***FCFS***. An arriving customer joins the queue of server 1 with probability $p_1 \in (0,1)$, server 2 with probility $p_2 \in (0,1)$ ... server n-1 with probability $p_{n-1} \in (0,1)$ and the queue of server n with probability $p_n = 1 - \sum_{i=1}^{n-1} p_i$. The service times at server 1 is $exp\ (\mu_i)$, server 2 is $exp\ (\mu_2)$ ... server $n-1$ is $exp\ (\mu_{n-1})$, and at server n is $exp(\mu_n)$ distributed. We assume the family of inter arrival times, service times and routing decisions to be an independent family of random variables.

Because we are intrested in steady state results, we assume that $\lambda p_1 < \mu_1, \lambda p_2 < \mu_2, \ldots, \lambda p_{n-1} < \mu_{n-1}$ *and* $\lambda p_n < \mu_n$.

A Markovian description of the systems development over time is given by recording the joint queue length process at server $1,2, \ldots, n-1$ and n. We assume that the associated Markov process is stationary with steady state distribution.

$$\Pi(m_1, m_2, \ldots, m_{n-1}, m_n) = \prod_{i=1}^{n}\left(1-\frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} \prod_{i=1}^{n-1}\left(1-\frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i}$$

$$(m_1, m_2, \ldots, m_{n-1}, m_n) \in N^n$$

Theorem.

$$f(s) = \prod_{i=1}^{n}\left[\frac{\mu_i - a\lambda p_i}{\mu_i - a\lambda p_i\left[\frac{\mu_i}{\mu_i+\mu_{i+1}}\right]}\,\frac{\mu_{i+1} - a\lambda p_{i+1}}{\mu_{i+1} - a\lambda p_{i+1}\left[\frac{\mu_{i+1}}{\mu_i+\mu_{i+1}}\right]s}\left(1+\frac{s}{\mu_i+\mu_{i+1}}\right)\right]$$

$$-\prod_{i=1}^{n}\left[\frac{s}{\mu_i+\mu_{i+1}}\left\{p_{i+1}\frac{\mu_i - a\lambda p_i}{\mu_{i+1} - a\lambda p_{i+1} + \left[\frac{\mu_n}{\mu_i+\mu_{i+1}}\right]s}\left(1-s\frac{a\lambda p_{i+1}}{\mu_{i+1}-a\lambda p_{i+1}}\right)\right.\right.$$

$$\left(a\lambda p_i + \mu_i + \mu_{i+1} - \frac{\mu_{i+1}}{\mu_{i+1}+a\lambda p_{i+1}}s - \frac{1}{\theta(\mu_{i+1} p_{i+1} a\lambda p_i \mu_i)}\right)^{-1}$$

$$+\prod_{i=1}^{n} p_i\frac{\mu_i - a\lambda p_{i+1}}{\mu_i - a\lambda p_i + \left[\frac{\mu_i}{\mu_i+\mu_{i+1}}\right]}\left(1-s\frac{a\lambda p_{i+1}}{\mu_i - a\lambda p_i}\right)$$

$$\left.\left.\left(a\lambda p_{i+1} + \mu_i + \mu_{i+1} - \frac{\mu_i}{\mu_i + a\lambda p_i}s - \frac{1}{\theta(\mu_i, p_i, \mu_i, s)}\right)^{-1}\right\}\right]$$

where $\theta\,(\eta, r, w, s) = \frac{A}{2a\lambda rw}\eta > 0,\ r \in [0,1],\ w \geq 0,\ s \geq 0$

where $A = \eta \dfrac{(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s\,)}{-\sqrt{\eta^2\,(\lambda\,(1-2r)+\mu_i+\mu_{i+1}+s\,)^2 - 4a\lambda p_i p_{i+1}\,\mu_i\mu_{i+1}}}$.

Proof. We introduce a sequence of conditional *LSTs* associated with the departure proce ss from the system.

Suppose that the system jumps any transition into the state

$(m_1, m_2, \dots, m_{n-1}, m_n) \in N^2 - [0,0]$

and a clock is started at this jump instant. For some $k \in [1, 2, \dots m_1 + m_2, \dots, m_{n-1} + m_n]$. Let the clock be stopped at the k^{th} departure instant after the clock is started and denoted by

$f\,(k\,; m_1, m_2, \dots, m_{n-1}, m_n)\,(s)\,,\ s \geq 0.$

The *LST* of the distribution of the time recorded by that clock. Then

$$f(s) = \Sigma\pi(m_1, m_2, \dots, m_n)\{p_1 f(m_1+m_2+1, m_1+1, m_2)(s) + p_2\, f(m_1+m_2+1, m_2+1)(s)\}, \dots, + p_n\, f(m_1, m_2, \dots, m_{n-1}, m_n, m_{n+1} = m_1)(s)\},\ s \geq 0 \quad \dots(1)$$

The key to our analysis is the following sequence of first entrance equation for the conditional *LSTs*. For

$1 \leq k \leq m_1 + m_2 + \dots + m_n$.

$$f(k; m_1, m_2, \dots, m_n)\,(s) = \frac{\pi(m_1, m_2, \dots, m_n)}{\pi(m_1, m_2, \dots, m_n)+s}\left[\frac{a\lambda\, p_1}{\pi(m_1, m_2)}\right.$$

$$f\,(k; m_1+1, m_2)(s) + \dots + \dots + \frac{a\lambda\, p_n}{\pi(m_1)} f(k; m_n+1, m_1)(s) + I_{m_{1>0}} \frac{\mu_1}{\lambda(m_1, m_2)}$$

$$f\,(k-1, m_2-1, m_2)(s) + I_{m_{2>0}} \frac{\mu_2}{\pi\ m_2, m_3} f\,(k-1, m_2-1, m_3)(s) + \dots +$$

$$\left. I_{m_{n>0}}\ \frac{\mu_n}{\pi\,(m_n, m_1)} f\,(k-1, m_n, m_1-1\,)(s)\right]\,,\ s \geq 0 \quad \dots(2a)$$

where $\lambda\,(m_1, m_2, \dots, m_n) = a\lambda + I_{m_{1>0}}\,\mu_1 + \dots + I_{(m_{n>0})}\,\mu_n$ and for

$$m_1 + m_2 + m_3 + \dots + m_n \geq 0,\ f(0, m_1, m_2, \dots, m_n)(s) = 1,\ s \geq 0 \quad \dots(2b)$$

We shall concentrate on cmputing in first step

$$F(0,s) = \left(\sum_{m_1=0}^{\infty} \dots \sum_{m_n=0}^{\infty}\ \prod_{i=1}^{m}\ (\frac{a\lambda p_i}{\mu_i})^{m_i} \,.\, f\,(m_1 + m_2 + \dots + m_n)(s),\ s \geq 0 \quad \dots 3\right.$$

which by a symmetry argument will provide (1). It turns out that we have to compute a recursion for the sequence

$F(h,s)\,;\ h = 0,1,\dots,\ s \geq 0$

to eventually obtain $F(0,s)$, *where*

$$F(h,s) = \sum_{m_1=h_1}^{\infty} \dots \sum_{m_n=h_n}^{\infty} \prod_{i=1}^{m} (\frac{a\lambda p_i}{\mu_i})^{m_i} \,.\, f\,(\,m_1 + m_2 + \dots + m_n + 1 - h_i, m_i + 1, \dots, m_n\,)s,$$

$$s \geq 0,\ h_i = 0, 1 \dots \quad \dots(4)$$

For the given $h_i \in n$, $m_i \in h_i, \dots, m_{n-1} \in h_i, m_n \geq 0$ and

$K = m_1 + m_2 +,\dots, m_n - h_i$

we multiply (2a) *(with $m_1 + 1$, $m_2 + 1$,..., instead of $m_1, m_2, \dots, +m_n$) by*

$(\Lambda \ (m_1 + 1, m_2 + 1, \dots, m_{n-1}, m_n) + s) \prod_{i=1}^{n} \left(\frac{\lambda p_i}{\mu_i}\right)^{m_i}$

and summing over $m = h_i, h_i + 1, \dots, m_n = 0, 1, \dots,$

we obtain, after some manipulations

$$F(h_1,s) = F(h_1+1,s)\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s} + \frac{a\lambda p_i}{\mu_1}\ \frac{\mu_1}{(\mu_1+\mu_2+s)}$$
$$+ (D(h_1,s) + D(h_1+1;s)\ \left(\frac{\mu_2}{a\lambda p_1}\right)\frac{\mu_1}{(\mu_1+\mu_2+s)}$$
$$+ (c(h_1,s) + c(h_1+1;s)\ \frac{\mu_1}{\mu_1+\mu_2+s}, s \geq 0, h_1 \in \mathrm{N}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$F(h_n,s) = F(h_n+1;s)\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s} + \frac{a\lambda p_n}{\mu_n}\frac{\mu_n}{(\mu_1+\mu_n+s)}$$
$$+ (D(h_n,s) + D(h_n+1;s)\left(\frac{\mu_n}{a\lambda p_n}\right)\frac{\mu_n}{(\mu_n+\mu_1+s)}$$
$$+ (c(h_n,s) + c(h_n+1,s)\ \frac{\mu_n}{\mu_n+\mu_1+s}, s \geq 0, h_1 \in \mathrm{N}.$$

We can write the above expression in general terms

$$F(h_i;s) = F(h_i+1;s)\ \frac{\mu_i+\mu_{i+1}}{\mu_i+\mu_{i+1}+s}\ \frac{\alpha\lambda p_i}{\mu_i}\frac{\mu_i}{(\mu_i+\mu_{i+1}+s)}$$
$$+ (D(h_i,s) + D(h_i+1;s)\ \left(\frac{\mu_i}{a\lambda p_n}\right)\frac{\mu_i}{(\mu_i+\mu_{i+1}+s)}$$
$$+ (c(h_i,s) - c(h_i+1;s), \frac{\mu_i}{\mu_i+\mu_{i+1}+s}, s \geq 0, h_1 \in \mathrm{N}$$

where

$$C(h_1;s) = \sum_{m_1=h_1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1} f(m_1+1-h_1;m_1+1,0)(s), s \geq 0, h_1 \in \mathrm{N}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$C(h_n;s) = \sum_{m_n=h_n}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1-h_n;m_n+1,0)(s), s \geq 0, h_n \in \mathrm{N}$$
$$C(h_i;s) = \sum_{m_1=h_1}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_n} f(m_i+1-h_i;m_i+1,0)(s), s \geq 0, h_i \in \mathrm{N}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$D(h_i;s) = \sum_{m_1=1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{h_1}\left(\frac{a\lambda p_2}{\mu_2}\right)^{m_2} f(m_1,h_1;m_2)(s), s \geq 0, h_1 \in \mathrm{N}$$
$$D(h_i;s) = \sum_{m_n=1}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n,h_n;m_n)(s), s \geq 0, h_n \in \mathrm{N}$$
$$D(h_i;s) = \sum_{m_i=1}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i,h_i;m_i)(s), s \geq 0, h_i \in \mathrm{N}.$$

Evaluating the generating functions $\sum_{h_n=1}^{\infty}(h_n,s)z^{h_n}, |z|<1$, for any $s\geq 0$, at $z=\left(\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}\right)$

we eventually obtain

$$F(0,s)=\frac{\mu}{\mu_1(\mu_1+\mu_2+s)-a\lambda p_1(\mu_1+\mu_2)}$$

$$+c(0,s)\frac{\mu_2}{\mu_1+\mu_2}-D(0,s)\frac{\mu_1\mu_2}{a\lambda_1 p_1(\mu_1+\mu_2)}$$

$$-\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}-c(h,s)\frac{\mu_2}{(\mu_1+\mu_2)(\mu_1+\mu_2+s)}$$

$$+\sum_{h_1=0}^{\infty}\left(\frac{\mu_1+\mu_2}{\mu_1+\mu_2+s}\right)^{h_1}D(h_1,s)\frac{\mu_2(\mu_1+\mu_2+s)+a\lambda p_1(\mu_1+\mu_2)}{a\lambda p_1(\mu_1+\mu_2)(\mu_1+\mu_2+s)},\ s>0.$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$F(0,s)=\frac{\mu_n^2}{\mu_n^2(\mu_n+\mu_1+s)-a\lambda p_1(\mu_n+\mu_1)}+C(0,s)\frac{\mu_n}{\mu_n+\mu_1}$$

$$-D(0,s)\frac{\mu_n\,\mu_1}{a\lambda p_n(\mu_n+\mu_1)}-\sum_{h_n=0}^{\infty}\frac{\mu_n\,\mu_1}{a\lambda p_n(\mu_1+\mu_2)}$$

$$-\sum_{h_n=0}^{\infty}\frac{\mu_n+\mu_1}{\mu_n+\mu_1+s}-D(h_n,s)\frac{\mu_2(\mu_2+\mu_1+s)+a\lambda p_1(\mu_n+\mu_1)}{a\lambda p_n(\mu_1+\mu_n)(\mu_1+\mu_n+s)}.$$

Finally, general expression can be established as

$$F(0,s)=\frac{\mu_i^2}{\mu_i(\mu_i+\mu_{i+1}+s)-a\lambda p_1(\mu_i+\mu_{i+1})}-$$

$$+C(0,s)\frac{\mu_{i+1}}{\mu_i+\mu_{i+1}}-D(0,s)\frac{\mu_i\,\mu_{1+1}}{a\lambda p_i(\mu_1+\mu_{i+1})}$$

$$-\sum_{h_i=0}^{\infty}\left(\frac{\mu_i+\mu_{i+1}}{\mu_i+\mu_{i+1}+s}\right)^{h_i}-D(h_i,s)\frac{\mu_i(\mu_i+\mu_{i+1}+s)+a\lambda p_n(\mu_i+\mu_{i+1}+s)\,\mu_i}{a\lambda p_i(\mu_i+\mu_{i+1})(\mu_i+\mu_{i+1}+s)} \quad \text{...(5)}$$

It remains to compute the boundary terms. This will be done using the observations that the following equations hold

$$f(k;k+m_1,m_2)(s)=f(k;k,m_2)(s),\ k,m_1,m_2\in N \quad \text{...(6a)}$$
$$f(k;m_1,k+m_2)(s)=f(k;m_1,k)(s),\ k,m_1,m_2\in N \quad \text{...(6b)}$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$f(k;k+m_{n-1},m_n)(s)=f(k;k,m_n)(s),\ k,m_{n-1},m_n\in N \quad \text{...(6c)}$$
$$f(k;m_{n-1},k+m_n)(s)=f(k;m_{n-1},k)(s),\ k,m_{n-1},m_n\in N \quad \text{...(6d)}$$
$$f(k;k+m_n,m)(s)=f(k;k,m_{n-1})(s),\ m_n,m_n\in N \quad \text{...(6e)}$$
$$f(k;m_n,k+m_1)(s)=f(k;m_n,k)(s),\ k,m_n,m_1\in N \quad \text{...(6f)}$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$f(k;m_i,k+m_{i+1})(s)=f(k;m_i,k)(s),\ k,m_i,m_{i+1}\in N \quad \text{...(6g)}$$

These equations are easily interpreted. In the time for the next departures from system that to be computed, we can neglect queuing customers that set atleast k customers in front of themselves (in their own queue) because after the departure of these k customers the requested time has already passed.

We introduce,

$$B(j,s) = \sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_1}{\mu_n}\right)^{m_1} f(m_1+1,m_1+1,j)(s); s \geq 0,\ j\in N$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$B(j_n,s) = \sum_{m_n=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1,m_n+1,j_n)(s); s \geq 0,\ j\in N$$

We can write the general expression of the above equations

$$B(j_i,s) = \sum_{m_i=0}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1,m_i+1,j_i)(s_i); s \geq 0,\ j_i\in N$$

Now,

$$C(h_1,s) = \sum_{m_1=h_1}^{\infty}\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1} f(m_1+1-h_1,m_1+1-h_1,0)(s); s \geq 0,\ h_1\in N$$

$$= B(0,s)\left(\frac{a\lambda p_i}{\mu_i}\right), s \geq 0\ h_1 \in N.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$C(h_n,s) = \sum_{m_1=0}^{\infty}\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n} f(m_n+1-h_n,m_n+1-h_n,0)(s); s \geq 0,\ h_n\in N.$$

Thus general expression from equations (6a) (6g) may be obtained as

$$C(h_i,s) = \sum_{m_i=h_i}^{\infty}\left(\frac{a\lambda p_i}{\mu_i}\right)^{m_i} f(m_i+1-h_i,m_i+1-h_i,0)(s); s \geq 0,\ h_i\in N.$$

To obtain $B(0,s)$ note that for the conditional $LSTs$ introduces (2a) reduces

$$f(m_1+1,m_1+1,j_1)(s)\,[a\lambda p_2+\mu+I_{(j>0)}\mu_2+s]$$
$$= f(m_1+1,m_1+1,j_1+1)(s)a\lambda p_2+f(m_1,m_1,j)(s)(\mu_1)+f(m_1,m_1,j-1)(a\lambda)I_{(j_1>0)}$$
$$j\in N,\ m_1\in N,\ s\geq 0. \qquad \ldots(7)$$

$$f(m_n+1,m_n+1,j_n+1)(s)\,[a\lambda(p_1)+\mu_n+I_{(j_n>0)}\mu_1+s]$$
$$= f(m_n+1,m_n+1)(s)\,a\lambda(p_n+1)$$
$$+ f(m_n,m_n,j_n)(s)(\mu_n)+f(m_n,m_n,j_n-1)(\lambda)((\mu_{n+1}z\mu_1)I_{(j_n>0)}$$
$$j_n\in N,\ m_n\in N,\ s\geq 0 \qquad \ldots(8)$$

Multiplication of (7) and (8) by $\left(\frac{a\lambda p_1}{\mu_1}\right)^{m_1}$ and $\left(\frac{a\lambda p_n}{\mu_n}\right)^{m_n}$ respectively and summation over m_1 and $m_n\in N$ yeilds the $B(j_n,s)$ respectively.

Evaluating the generating functions

$$\sum_{j_1=0} z^{j_1} B(j_1,s), |z| < 1, s \geq 0$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$\sum_{j_n=0} z^{j_n} B(j_n,s), |z| < 1, s \geq 0$$

at $\theta(\mu_1, p_1, \mu_2, s)$ and $\theta_1(\mu_n, p_n, \mu_1, s)$ the smaller roots of the equation

$$a\lambda p_1 \mu_2 z \ldots \mu_1(\lambda(1-2p_1) + \mu_1 +$$
$$\mu_2 + s)z + a\lambda p_2 \mu_1 = 0$$
$$a\lambda p_{n-1}\mu_n z \ldots \mu_n(\lambda(1-2p_n) + \mu_n, \mu_1, s)z + a\lambda p_1 \mu_n = 0$$

We obtain general expression as under

$$B(0,s) = \frac{\theta(\mu_i + p_i + \mu_{i+1}, s)}{1 - \theta(\mu_i, p_i, \mu_{i+1}, s)} \quad \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{a\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1} + s)}$$

and

$$C(0,s) \frac{\mu_{i+1}}{\mu_i + \mu_{i+1}} - \sum_{h_1=0}^{\infty} \left(\frac{\mu_i + \mu_{i+1}}{\mu_1 + \mu_{i+1} + s} \right)^h C(h,s) \frac{s\mu_{i+1}}{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$= \frac{\theta(\mu_i, p_i, \mu_{i+1}, s)}{1 - \theta(\mu_i, \mu_{i+1}, s)} \cdot \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\lambda p_{i+1} - \mu_{i+1} \; \theta(\mu_i, p_i, \mu_{i+1}, s)}$$

$$\mu_{i+1} \quad \frac{\mu_{i+1} + a\lambda p_i}{\mu_i(\mu_i + \mu_{i+1} + s) - a\lambda(\mu_i + \mu_{i+1})} \qquad \ldots(9)$$

To obtain $D(h,s)$, $h \in N$ note that for the conditinoal *LSTs* envolved, because of (6), (2a) reduces to

$$f(m_n, h, m_n)(s) \; [\lambda p_i + I_{(h>0)} \mu_i + \mu_i + s]$$
$$= f(m_i, h+1, m_i)(s)[a\lambda p_i + I_{(h>0)} \mu_i f(m_n - 1, h-1, m_n - 1]$$
$$+ \mu_n f(m_n - 1, h, m_n - 1)(s) \; s \geq 0, \; h \in N, \; m_n \geq 1. \qquad \ldots(10)$$

Multiplication of (6) by $\left(\frac{a\lambda p_i}{\mu_i}\right)^h \left(\frac{a\lambda p_{i+1}}{\mu_{i+1}}\right)^{m_n}$ and summation over

$m_n = 1, 2 \ldots$ yeilds

$$D(h,s)[a\lambda\,(1-2\,p_{i+1}) + 1_{(h>0)} \mu_i + \mu_{i+1} + s]$$
$$= \left(\frac{a\lambda p_i}{\mu_i}\right)^h a\lambda p_{i+1} (1 + I_{(h_i>0)} \frac{\mu_i}{\mu_{i+1}})$$
$$+ D(h_{i+1}, s)\, \mu_i$$
$$+ D(h-1,s) \frac{a\lambda^n p_i p_{i+1} I_{(h_1>0)}}{\mu_i}, s \geq 0, \; h \in N.$$

For generating function, we therfore have

$$\sum_{h=0}^{\infty} Z^h D(h,s) \left[(a\lambda(1-2p_{i+1}) + \mu_i + \mu_{i+1} + s) - \frac{\mu_i}{2} - \frac{a\lambda^2 p_i p_{i+1} z}{\mu_{i+1}} \right.$$

$$= D(0,s)\mu_i \; \left(\frac{z-1}{2}\right) + \frac{a\lambda^2 p_{i+1} \mu_i}{\mu_{i+1}} \cdot \frac{\mu_{i+1} + a\lambda p_i z}{\mu_i - a\lambda p_i z},$$

$$|z| < 1, \; s \geq 0 \qquad \ldots(11)$$

With $\theta(\mu_{i+1}, p_i, a\lambda p_i, s)$ the smaller root of the equation

$$p_i p_{i+1} - z^n - \mu_{i+1}(a\lambda(1-2p_{i+1}) + \mu_i + \mu_{i+1} + s)z + \mu_{i+1}\mu_i = 0,$$

we obtain from (11)

$$F(0,s) = \frac{a\lambda p_{i+1}}{\mu_{i+1}} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{\mu_{i+1} + a\lambda p_i}{\mu_i} - \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

and

$$\sum_{h=0}^{\infty} \left(\frac{\mu_i + \mu_{i+1}}{\mu_i + \mu_{i+1} + s}\right)^h D(h,s) \; \frac{\mu_i(\mu_i + \mu_{i+1} + s) + a\lambda p_i(\mu_i + \mu_{i+1})}{a\lambda p_i(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)}$$

$$\mu_i - D(0,s)\frac{\mu_i + \mu_{i+1}}{a\lambda p_i, (\mu_i + \mu_{i+1})}$$

$$= -\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + a\lambda p_i \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i(\mu_i + \mu_{i+1})}\left[1 + \frac{s\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}\right] + \frac{p_{i+1}\mu^2}{p_i}$$

$$\frac{\mu_i + (\mu_i + \mu_{i+1})(\mu_i + a\lambda p_i)}{(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)(\mu_i s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})} \quad \text{...(12)}$$

Inserting (7) and (12) into (5) we finally obtain

$$F(0,s) = \frac{\mu_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_i)}$$

$$\left(1 + \frac{p_{i+1}}{\mu_i} \cdot \frac{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_{i+1} + a\lambda p_i)}\right)$$

$$\frac{-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \; \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\frac{p_{i+1}\mu_i}{p_i} \; \frac{\mu_{i+1} - a\lambda p_{i+1} + s}{\mu_n s + (\mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$+ \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \; \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)}{a\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)} \quad \text{...(13)}$$

$$\mu_{i+1} \; \frac{\mu_i - a\lambda p_i}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)}$$

The transformation

$$\begin{pmatrix} \mu_i \\ \mu_{i+1} \\ p_i \end{pmatrix} \xrightarrow{T} \begin{pmatrix} \mu_i \\ \mu_{i+1} \\ p_i \end{pmatrix}$$

$$\frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_{i+1}, a\lambda p_i, s)}$$

$$\xrightarrow{T} \frac{\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + \alpha\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - \alpha\lambda p_i - \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}$$

Therefore, applying T to $F(0,s)$ and inserting the results and $F(0,s)$ into (1) yeilds

$$F(0,s) = \frac{(\mu_i + \mu_{i+1})(\mu_i + \mu_{i+1} + s)(\mu_i - a\lambda p_i)(\mu_{i+1} - a\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - a\lambda p_i)(\mu_{i+1} s + \mu_i + \mu_{i+1})(\mu_{i+1} - a\lambda p_{i+1})}$$

$$- s\left\{\frac{\theta(\mu_i, p_{i+1}, a\lambda p_i, s)}{1-\theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)} \frac{\mu_i + a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}{\mu_i - a\lambda p_i . \theta(\mu_{i+1}, p_{i+1}, a\lambda p_i, s)}\right\}$$

$$\frac{1}{\mu_i} \frac{p_{i+1} \qquad (\mu_i - a\lambda p_i) \qquad (\mu_{i+1} - a\lambda p_{i+1})}{\mu_{i+1} s \quad + (\mu_i + \mu_{i+1}) \qquad (\mu_i - a\lambda p_{i+1})}$$

$$+ \frac{\theta(\mu_i, p_i, \mu_i, s)}{1-\theta(\mu_i, p_i, \mu_i, s)} \frac{\mu_i + \mu_{i+1} . \theta(\mu_i, p_i, \mu_{i+1}, s)}{\alpha\lambda p_{i+1} - \mu_{i+1} . \theta(\mu_i, p_i, \mu_i, s)}$$

$$\frac{1}{\mu_i} \frac{p_i \ (\mu_i - \alpha\lambda p_i)(\mu_{i+1} - \alpha\lambda p_{i+1})}{\mu_i s + (\mu_i + \mu_{i+1})(\mu_i - \alpha\lambda p_i)}.$$

After sum direct input this yield the theroem.

3. Probability Distribution of Working and Non Working States of the System. Assumptions

We analyse the probability distirbution of the system when it is in working and non-working state as under:

Following assumptions are laid down before we compute the transition probabilities from working state of the system to non-working, vice versa and when system remains at itself.

(i) Servers are parallelly arranged.

(ii) Servers are simulteneously busy and working as one single system.

(iii) System may go in non-working state at any epoch of the time and assumed to come back in normal state i.e.; working after certain time period. Reasons of non-working may be such as idleness, failure or any type of distribution etc.

(iv) Let working state of the system be represented by 1 and non working state be represented by *0*.

3.1 Transitional Probability distribution. Let the lengths of

working and non-working periods be independent random variables having negative exponential distribution with means $1/b$ and $1/c$ respectively $(b,c > 0)$.

Now

$$q_{01}(\Delta t) = Pr\,\{change\ of\ state\ from\ 0\ to\ 1\ in\ time\ \Delta t\} = b\,\Delta t + o\,(\Delta t)$$

$$q_{10}(\Delta t) = Pr\,\{change\ of\ state\ from\ 1\ to\ 0\ in\ time\ \Delta t\} = c\,\Delta t + o\,(\Delta t).$$

Transition probabilities are

$$b_{01} = b, \quad b_{00} = -b \text{ and } b_{10} = c, \quad b_{11} = -c$$

Now the transition probabilities martix may be written as

$$A = \begin{bmatrix} -b & b \\ c & -c \end{bmatrix}$$

The Kolmogrov forward equations for $i = 0, 1$ are

$$q'_{i_0}(t) = -b\,q_{i_0}(t) + cq_{i_0}(t)$$
$$q'_{i_1}(t) = b\,q_{i_0}(t) - cq_{i_1}(t)$$

We use the following equations while finding the transition probabilities $q_{i_1}(t)$:

$$q_{00}(t) - q_{01}(t) = 1$$
$$q_{10}(t) + q_{11}(t) = 1.$$

We have

$$q'_{00}(t) + (b-c)\,q_{00}(t) = c$$

and

$$q'_{11}(t) + (b+c)\,q_{11}(t) = b$$

The solution of the frist of these differential equations is

$$q_{00}(t) = \frac{c}{b+c} + Ce^{-(b+c)t}$$

with $q_{00}(0) = 1$, we find $C = \frac{c}{b+c}$ so that

$$q_{00}(t) = \frac{c}{b+c} + \frac{c}{b+c}\,e^{-(b+c)t}$$

Hence

$$q_{01}(t) = 1 - q_{00}\frac{b}{b+c} + \frac{b}{b+c}\ ^{-(b+c)t}$$

Proceeding exactly in the same way, the solution of the second differential equation with the initial condition $q_{11}(t) = 1$, yelids

$$q_{11}(t) = \frac{b}{b+c} + \frac{b}{b+c}\,e^{-(b+c)t}$$

and therefore

$$q_{11}(t) = 1 - q_{11}(t)$$

$$= \frac{c}{b+c} - \frac{c}{b+c} e^{-(b+c)t}$$

We can finally conclude that the present work not only envisages the multistate space with working and non-working states of the system but also it respeakes more general and elegant unification of the results including Daduna [3] and others. The results are obtainable in turn, after specializing various parameters such as $i = 1,2$; $m_i = m_1$, $m_2 = n$, $E(x) = 1$ and non-working state is assumed not to occur with the system.

REFERENCES

[1] M. Berg and M.J.M. Posner, Customer Delay in *M/M/C* Repair System with spares, *Naval Reslogist Quart.* **32** (1985), 287-299.

[2] M. Berg and M.J.M. Posner, On the Regulation of Queues of Lelt 4 (1986) 221-224.

[3] H. Daduna, Exchangeble items in Repair systems; *Delay Times, Operation Research* (1989) 349-354.

[4] R.L. Disney and D. Konig, Queueing Networks; A Survwy of their Random Process, *Siam Rev.* **27** (1985), 335-403.

[5] R. Schass Berger, *Exact Results on Response time Distribution in Networks of Queues.* In Beilher *H.* (ed). Messung, Modellerung and Bewerluing Von Rechensystemen, 1985.

[6] H. Schroder, Eine Spexielle Methods Zir Fertifufssouring beider Reparates Von Flugzengu Leufteien, *In Oshe Det al. Operation Research Proceeding*, DGOR, Springer, Berlin, (1985), *P. 239.*

Jñānābha, Vol. 30, 2000

A RESULT ON SIMPLE ACCESSIBLE RINGS

By

K. Suvarna and K. Subhashini

Department of Mathematics, S.K. University,
Anatapur-515 003, Andhra Pradesh, India

(Received : July 10, 1999)

ABSTRACT

In this paper we show that a simple accessible ring is either a $(-1,1)$ ring or a commutative ring.

1. Introduction. Accessible rings were introduced by Kleinfeld [1]. A non-associative ring R is called accessible in case the following two identities hold :

(1) $(x,y,z) + (z,x,y) - (x,z,y) = 0,$

(2) $((w,x), y,z) = 0$

for all w,x,y,z in R, where the associator (x,y,z) is defined by $(x,y,z) = (xy) z - x(yz)$ and the commutator (x,y) is defined by $(x,y) = xy - yx$, An accessible ring is defined to be simple if it has no proper two sided ideals.

A $(-1,1)$ ring is a non-associative ring in which the following identities hold :

(3) $(x,y,z) + (x,z,y) = 0$,

(4) $(x,y,z) + (y,z,x) + (z,x,y) = 0.$

In [4] it is proved that a simple non-associative ring of char $\neq 2$ satisfying the identity $((x,y,z),w) = 0$ is either commutative or associative, Kleinfeld [2] proved that the same result still valid when this identity is replaced by more general conditions. The identity $((x,y,z)\ v,w) = 0$ holds in accessible rings under the assumption that the rings are without nilpotent elements in the center [1]. Without this assumption we show that $((x,y,y)\ v,w) = 0$ holds in accessible rings. Using this identity we show that a simple accessible ring is either $(-1,1)$ or commutative.

2. Preliminaries. By substituting $z = x$ in (1) we obtain the flexible law $(x,y,x) = 0$. The following identities hold in accessible rings

(5) $(x,y,z) = - (z,y,x)$,

(6) $(x,y,z) = x\,(y,z) + (x,z)\,y,$

(7) $(x,y,z) + (y,z,x) + (z,x,y) = 0,$

(8) $((w,x,y),z) = 0$.

The nucleus N of R is defined as the set of all elements n in R with the property $(n,R,R) = 0$, If n is an element of the nucleus N of R, then because

of the flexible law $(R,R,n) = 0$, Finally because of (1) it follows that also $(R,n,R) = 0$.

The following identity holds in an arbitrary ring :

$(wx,y,z) -(w,xy,z)+ (w,x,yz) = w(x,y,z) + (w,x,y)\, z.$

If $w = n$ in the above equation, then it becomes

(9) $(nx,y,z) = n(x,y,z)$, n in N.

From (9) and from the fact that every commutator is in the nucleus, we get $\quad (v,x)\,(x,y,z) = ((v,x)x,y,z)$.

It follows from (6) that $(v,x)\, x= (v,x,x)$. Consequently

$((v,x)x,y,z) = ((vx,x),y,z) = 0$, Thus

(10) $\quad (v,x)(x,y,z) = 0.$

3. Main Result

Lemma 1. In an accessible ring $R,((x,y,y)v,w) = 0$.

Proof. Linearization of (10) becomes

(11) $\quad (v,w)\,(x,y,z) = -(v,x)\,(w,y,z)$

By using (5), (11), (7) and (10),

$= (v,y)\,(w,y,x)$

$= (v,y)\,[-(y,x,w) - (x,w,y)]$

$= - (v,y)\,(y,x,w) + (v,y)\,(y,w,x)$

$= 0.$

That is,

(12) $\quad (v,w)\,(x,y,y) = 0.$

Now from(6),(12)and(8), we get $((x,y,y)v,w)=(x,y,y)(v,w)+((x,y,y),w)v= 0$

Lemma 2. Let R be an accessible ring, then $U = \{u\in R/(u,r) = 0 = (uR,R)\}$ is an ideal of R.

Proof. If we put $w = u$ in (8), then $((u,x,y),z) = 0$. From this it follows that $(ux.y,z)-(u.xy,z) = 0$. Then $(ux,y,z) = 0$ by the definition of U. Thus $ux \in U$. So U is a right ideal of R. Since $(u,R) = 0$, $(u,x) = 0$. That is $ux = xu\in U$. So U is a left ideal of R. Hence U is an Ideal of R.

Theorem. If R is a simple accessible ring, then R is either a $(-1,1)$ ring or a commutative ring.

Proof. From (8) and lemma 1 (x,y,y) is in U. Since U is an ideal of R and R is simple, we have either $U=0$ or $U = R$. If $U = 0$, then R is right alternative, that is $(x,y,y) = 0$.

Linearization of this yields $(x,y,z) + (x,z,y) = 0$. From this and (7) it follows that R is a $(-1,1)$ ring. If $U = R$ then R is a commutative ring.

REFERENCES

[1] E.Kleinfeld, Standard and accessible rings, *Canad. J. Math.* 8 (1956), 335-340.

[2] E. Kleinfeld, On rings with alternators in the commutative center, *Port. Math.* 50 (1993), 297-304

[3] Yash Paul, A note on (–1,1) rings, *Jñānābha* 11 (1981), 107–109

[4] A Thedy, On rings satisfying $((a,b,c)\,,d) = 0$, *Proc. Amer. Math. Soc.* 29 (1971), 213-218.

Jñānābha, Vol. 30, 2000

SOME OPERATION- TRANSFORM FORMULAE FOR S_μ- TRANSFORM

By

M.M. P. Singh

Department of Mathematics, Ranchi University, Ranchi-8, Bihar

(Received : July 20, 1999)

1. Introduction. In an earlier Paper [1], S_μ- transform has been defined by

(1.1) $$S_\mu [f(t)] = F(s) = \int_0^\infty f(t) \frac{e^{-\mu(s/t)}}{s+t} dt, \ (\mu \geq 0),$$

where $f(t)$ is a suitably restricted conventional function defined on the real line $0 < t < \infty$ and $0 < Re(S) < \infty$. It has been generalised in the case of generalised functions as

(1.2) $$S_\mu [f(t)] = F(s) = < f(t), \frac{e^{-\mu(s/t)}}{s+t} >, \ (\mu \geq 0).$$

Its inversion formula has been also derived. Here it is proposed to discuss some operation-transform formulae of the transform given by (1.1).

2. Operation- Transform Formulae for S_μ- Tranform

Differentation : If $\phi \in B_\mu$, where B_μ is the space of all complex valued smooth functions $\phi(t)$ such that for each $\phi(t) \in B_\mu$, we have

$$\rho_n(\phi) = \sup_{0<t<\infty} |D^n \phi(t)| \qquad (n = 0,1,2,...)$$

bounded.

We shall prove that

(2.1) $\rho_n [-D\phi] = \rho_{n+1} [\phi]$.

Since,

$$\rho_n [-D\phi] = \sup_{0<t<\infty} |D^n(-D\phi)|$$
$$\doteq \sup_{0<t<\infty} |D^{n+1} \phi|$$
$$= \rho_{n+1} |\phi|.$$

Therefore, we get

$$\rho_n [-D\phi] = \rho_{n+1} [\phi]$$

From (2.1) it follows that $\phi \to -D\phi$ is a continuous and linear mapping of B_μ on to itself. Therefore, from Theorem 1. 10–1 due to Zemanien [2.

p. 29], the adjoint mapping $f \to Df$ is also a continuous and linear mapping of β'_{μ} on to itself where B'_{μ} is the dual of B_{μ} and therefore we get

(2.2) $< Df(t), \phi(t) > = < f(t), -D\phi(t) >$

Now, we prove that the following operation-transform formula

(2.3) $S_{\mu}[D^n f] \le K.S_{\mu}[|f(t)|]$

Proof. Using, the generalised definition of S_{μ}-transform and the relation (2.2), we get

$$S_{\mu}[D^n f] = < D^n f(t), \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), (-D)^n \frac{e^{-\mu(s/t)}}{s+t} >$$

$$= < f(t), \sum_{\upsilon=0}^{n} {}^n c_{\upsilon} (-D)^{n-\upsilon} e^{-\mu\, s/t} (-D)^{\nu} \frac{1}{s+t} >$$

Therefore, we get

(2.4) $$S_{\mu}[D^n f(t)] = < f(t), \frac{e^{-\mu(s/t)}}{s+t} \cdot \frac{P_n(t)}{Q_n(t)} >$$

where $P_n(t)$ and $Q_n(t)$ are the polynomials in t such that order of $Q_n(t) \ge$ order of $P_n(t)$.

Let us suppose that f is a regular generalised function of B'_{μ}. Therefore, for, $\phi \in \beta_{\mu}$, we have

$$< f, \phi > = \int_0^{\infty} f(t)\,\phi(t)\,dt$$

and

$$|< f, \phi >| \le \int_0^{\infty} |f(t)|\,|\phi(t)|\,dt$$

Consequently, we get

(2.5) $|< f, \phi >| \le < |f|, |\phi| >$.

An appeal to (2.4) and (2.5) gives

$$|S_{\mu}[D^n f]| \le < |f(t), |-\frac{e^{-\mu s/t}}{s+t}|.|\frac{P_n(t)}{Q_n(t)}| >$$

$$\le < |f(t), |\frac{e^{-\mu s/t}}{s+t}|. K >,$$

where $|\frac{P_n(t)}{Q_n(t)}| \le K$ (const); $0 < t < \infty$; $0 < \mu.s < \infty$ and $\mu \ge 0$.

Therefore, we get

$$|S_{\mu}[D^n f]| \le K < |f(t), |\frac{e^{-\mu s/t}}{s+t}| >$$

$$\le K.\, S_{\mu}[|f(t)|].$$

This completes the proof.

Multiplication by an Exponential Function. Let μ be a real number such that $\mu \geq 0$. Now we prove that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and linear mapping from B_μ on to itself.

Proof. Let $\phi \in B_\mu$, We have

$$D^n[e^{-\mu t}\phi(t)] = \sum_{v=0}^{n} {}^nc_v\, D^{n-v} e^{-\mu t} D^v \phi(t)$$

$$= \sum_{v=0}^{n} {}^nc_v\, (-\mu)^{n-v} e^{-\mu t} D^v \phi(t).$$

Therefore, we get

$$|D^n[e^{-\mu t}\phi(t)]| \leq \sum_{v=0}^{n} K\,|D^v\phi(t)|$$

where $|\,{}^nc_v\,(-\mu)^{n-v} e^{-\mu t}| \leq K$ for $\mu \geq 0$ and $0 < t < \infty$.

Thus we get

(2.6) $\rho_n[e^{-\mu t}\phi(t)]\,K \leq \sum_{v=0}^{n} |\rho_v[\phi(t)]|$ $(n = 0,1,2,\ldots;\ v = 0,1,2,\ldots)$.

From (2.6), it follow that $\phi(t) \to e^{-\mu t}\phi(t)$ is a continuous and lineir mapping of B_μ on to itself, Therefore, from Theorem 1.10-1 due to Zemanian [2,p.29] the adjoint mapping $f \to e^{-\mu t} f$ is also a continuous and linear mapping of B'_μ on to itself and we get

(2.7) $< e^{-\mu t} f(t), \phi(t) > = < f(t), e^{-\mu t}\phi(t) >$.

An appeal to (2.7) and the generalised definition of S_μ-transform, we get

$$S_\mu[e^{-\mu t} f(t)] = < e^{-\mu t} f(t), \frac{e^{-\mu s/t}}{s+t} >$$

$$= < f(t), e^{-\mu t}\frac{e^{-\mu s/t}}{s+t} >.$$

Therefore,

$$|S_\mu[e^{-\mu t} f(t)]| \leq < |f(t)|, |e^{-\mu t}|.|\frac{e^{-\mu s/t}}{s+t}| >$$

by (2.5) if f is a regular generalised funciton

$$\leq \quad M < |f(t)|, \frac{e^{-\mu s/t}}{s+t} >$$

$$\leq \quad M\,S_\mu[\,|f(t)|\,]$$

where $|e^{-\mu t}| \leq M$

Thus we get an operation-transform formula

(2.8) $|S_\mu[e^{-\mu t} f(t)]| \leq M\,S_\mu[\,|f(t)|\,]$.

Multiplication by $(s+t)^{-\lambda}$ where $\lambda > 0$; $0 < t < \infty$ and $0 < s < \infty$.

We prove that $\phi(t) \to (s+t)^{-\lambda}\phi(t)$ is a continuous and linear mapping

of B_μ on to itself, where $\lambda > 0$; $0 < t < \infty$ and $0 < s < \infty$.

Proof. Let $\phi \in B_\mu$, we have

$$D_n[(s+t)^{-\lambda}\phi(t)] = \sum_{\upsilon=0}^{n} {}^nc_\upsilon D^{n-\upsilon}(s+t)^{-\lambda} D^\upsilon \phi(t)$$

$$= \sum_{\upsilon=0}^{n} {}^nc_\upsilon (-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon} D^\upsilon \phi(t).$$

Therefore, we get

$$|D_n[(s+t)^{-\lambda}\phi(t)]| \le M \sum_{\upsilon=0}^{n} |D^\upsilon \phi(t)|$$

where $|{}^nc_\upsilon (-\lambda)(-\lambda-1)\ldots(-\lambda-(n-\upsilon-1)(s+t)^{\lambda-\upsilon+\upsilon}| \le M$.

$(n = 0,1,2, \ldots. ; \upsilon = 0,1,2,\ldots)$

Thus, we get

(2.9) $\rho_n[(s+t)^{\lambda}\phi(t)] \le M \sum_{\upsilon=0}^{n} \rho_\upsilon[\phi(t)]$.

From (2.9) it follows that $\phi(t) \to (s+t)^{-\lambda}\phi(t)$ is a continuous and linear mapping of B_μ on to itself. Therefore, from the theorem 1.10–1 due to Zemanian [2.p 29]. the adjoint mapping $f \to (s+t)^{-\lambda} f$ of $\phi \to (s+t)^{-\lambda}\phi$ is also a continuous and linear mapping of B'_μ on itself and we get

(2.10) $< (s+t)^{-\lambda} f(t). \phi(t), \quad f(t).(s+t)^{-\lambda}\phi(t)>$.

An appeal to (2.10) and the generalised definition of S_μ -tranform gives

$$S_\mu[(s+t)^{-\lambda} f(t)| = < (s+t)^{-\lambda} f(t), e^{-\mu s/t} >$$

$$= < f(t), (s+t)^{-\lambda} \frac{e^{-\mu s/t}}{s+t} > .$$

If be a regular generalised function then by using (2.5), *we get*

$$|S_\mu[(s+t)^{-\lambda} f(t)]| \le < |f(t)|, |(s+t)^{-\lambda}| \,|\frac{e^{-\mu s/t}}{s+t}| >$$

$$\le N < |f(t)|, \frac{e^{-\mu s/t}}{s+t} > \quad \le N S_\mu[|f(t)|]$$

where $\qquad 0 \le |(s+t)^{-\lambda}| \le N$.

Thus we get an operation-transform formula

(2.11) $|S_\mu[(s+t)^{-\lambda} f(t)]| \le N S_\mu[|f(t)|]$.

Shifting. Let T be a fixed real number such that $0 < t+T < \infty$ and $0 < t < \infty$. Let $\phi(t) \in B_\mu$ Now we shall prove that $(t+T)$ is a continuous and linear mapping of B_μ on to itself.

Proof. Let us consider

$$D^n[\phi(t+T)] = (d/dt)^n |\phi(t+T)|$$

$$= [\frac{d}{d(1+T)} \frac{d(t+T)}{dt}]^n |\phi(t+T)|$$

$$= \left(\frac{d}{d(t+T)}\right)^n [\phi\,(t+T)]$$
$$= D^n_{t+T}\,\phi\,(t+T)$$
$$= D^n_{t_1}\,[\phi\, t_1],\ [t_1 = t+T]$$

where $0 < t + T < \infty$ and $0 < t < \infty$.

Therefore, we get

(2.12) $D^n_t\,[\phi(t+T)] = D^n_t\,[\phi(t)],\ [t_1 = t]$

i.e. $\rho_n\,[\phi(t+T)] = \rho_n\,[\phi(t)]$.

Thus from (2.12), it follows that $\phi\,(t) \to \phi(t+T)$ is a continuous and linear mapping of B_μ on to itself. Its inners sapping $\phi\,(t) \to \phi(t+T)$ is also a continuous and linear mapping of B_μ to on itself. Therefore $\phi\,(t) \to (t+T)$ is an isomorphiam of B_μ onto itself, the adjoint mapping of $\phi\,(t) \to \phi(t+T)$ is $f(t) \to f\,(t+T)$ which is also a continuous and linear mapping of B'_μ onto itself due to Theorem 1.10-1 of Zemanian [2,p. 29] and we get

(2.13) $< f\,(t+T), \phi\,(t) > = < f(t), \phi\,(t+T) >$

ACKNOWLEDGEMENT

I am grateful to Late Dr. K.M. Saksena, Rtd. Professor and Head Ranchi University for his guidance and help in the preparation of this paper.

REFERENCES

[1] M.M.P. Singh, A generalised Stieltjes transform S_μ of a class of generalised function, *Bull. Cal., Math Society*. **76** (1984), 295-300.

[2] A.H. Zemanian, Generalised Integral Transform, John Willey and Sons, Interscience Publishers, (1968).

Jñānābha, Vol. 30, 2000

FRACTIONAL INTEGRAL FORMULAE INVOLVING THE PRODUCT OF A GENERAL CLASS OF POLYNOMIALS AND *I*-FUNCTION OF TWO VARIABLES-II

By

Shailesh Jaloree, Anil Goyal and R.D. Agrawal
Department of Applied Mathematics
Samrat Ashok Technological Institute, Vidisha-464001, M.P.

(Received : November 15, 1999)

ABSTRACT

In the present paper we establish some fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables. Our results are quite general in character and a number of known and new formulae can be obtained as their particular cases. Several such intersting special and confluent cases of our main results are mentioned breifly.

1. Introduction. The object of the present paper is to obtain a few fractional integral formulae involving the product of a general class of polynomials and the *I*-function of two variables.

The fractional integral operator investigated by Erdélyi [1] and Kober [4] is defined as

$$I_x^{\eta,\upsilon}\{f(x)\}=\frac{x^{-\eta-\upsilon+1}}{\Gamma(\upsilon)}\int_0^x (x-t)^{\upsilon-1}t^{\eta-1}f(t)dt,\ Re(\upsilon)>0,\ \eta>0\,. \quad ...(1.1)$$

Since in the *I*-function of two variables we use the notation *I* therefore in the operator $I_x^{\eta,\upsilon}$, instead of *I* we shall use *U* and in place of η we shall use ς and therefore the operator $I_x^{(\eta,\upsilon)}$ will be represented by $U_x^{\varsigma,\upsilon}$.

The *I*-function of two variables defined by Goyal and Agrawal [2] in the following manner:

$$I^{m_1,n_1:m_2,n_2;m_3,n_3}_{p,q:p_i^{(1)},q_i^{(1)},p_i^{(2)},q_i^{(2)}:r}\left[\begin{matrix}Z_1\\Z_2\end{matrix}\middle|\begin{matrix}[(e_p:E_p,E'_p)]:(a_j,\alpha_j)_{1,n_2}],(a_{ji},\alpha_{ji})_{n_2+1,p_i^{(1)}}];[(c_j,\gamma_j)_{1,n_3}],\\ [(f_q:F_q,F'_q)]:(b_j,\beta_j)_{1,m_2}],(b_{ji},\beta_{ji})_{m_2+1,q_i^{(1)}}];(d_j,\delta_j)_{1,m_3}],\end{matrix}\right.$$
$$\left.\begin{matrix}(c_{ji},\gamma_{ji})_{n_3+1,p_i^{(2)}}]\\(d_{ji},\delta_{ji})_{m_3+1,q_i^{(2)}}]\end{matrix}\right]=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\,\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}\,d\xi d\eta \quad ...(1.2)$$

where

$$\Phi_1(\xi)=\frac{\prod_{j=1}^{m_2}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n_2}\Gamma(1-a_j+\alpha_j\xi)}{\sum_{j=1}^{r}\left[\prod_{j=m_2+1}^{q_i^{(1)}}\Gamma(1-b_{ji}+\beta_{ji}\xi)\prod_{j=n_2+1}^{p_i^{(1)}}\Gamma(a_{ji}-\alpha_{ji}\xi)\right]}. \quad ...(1.2.1)$$

$$\Phi_2(\eta) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j \eta) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j \eta)}{\sum_{i=1}^{r} \left[\prod_{j=m_3+1}^{q_i^{(2)}} \Gamma(1 - d_{ji} + \delta_{ji}\eta) \prod_{j=n_3+1}^{p_i^{(2)}} \Gamma(c_{ji} - \gamma_{ji}\eta) \right]}. \quad \text{... (1.2.2)}$$

$$\Psi(\xi,\eta) = \frac{\prod_{j=1}^{m_1} \Gamma(f_j - F_j\xi - F'_j\eta) \prod_{j=1}^{n_1} \Gamma(1 - e_j + E_j\xi + E'_j\eta)}{\prod_{j=m_1+1}^{q} \Gamma(1 - f_j + F_j\xi + F'_j\eta) \prod_{j=n_1+1}^{p} \Gamma(e_j - E_j\xi - E'_j\eta)}. \quad \text{... (1.2.3)}$$

Z_1, Z_2 are non zero complex variables; L_1, L_2 are two Mellin-Barne type contour integrals. Convergence conditions are

$$|arg\, z_1| < \frac{A_i \pi}{2}, \ |arg\, z_2| < \frac{B_i \pi}{2}$$

$$A_i = \sum_{j=1}^{n_1} E_j - \sum_{j=n_1+1}^{p} E_j + \sum_{j=1}^{m_1} F_j - \sum_{j=m_1+1}^{q} F_j + \sum_{j=1}^{m_2} \beta_j - \sum_{j=m_2+1}^{q_i^{(1)}} \beta_{ji} + \sum_{j=1}^{n_2} \alpha_j - \sum_{j=n_2+1}^{p_i^{(1)}} \alpha_{ji} > 0$$

and

$$B_i = \sum_{j=1}^{n_1} E'_j - \sum_{j=n_1+1}^{p} E'_j + \sum_{j=1}^{m_1} F'_j - \sum_{j=m_1+1}^{q} F'_j + \sum_{j=1}^{m_3} \delta_j - \sum_{j=m_3+1}^{q_i^{(2)}} \delta_{ji} + \sum_{j=1}^{n_3} \gamma_j - \sum_{j=n_3+1}^{p_i^{(2)}} \gamma_{ji} > 0,$$

for $i = 1, \ldots, r$.

A general class of polynomials $S_n^m[x]$ defined by Srivastava [8] as

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, x^k \qquad n = 0,1,2,\ldots \quad \text{...(1.3)}$$

where m is an arbitrary positive integer and $A_{n,k}$ $(n,k \geq 0)$ are arbitrary constants, real or complex.

On specializing the coefficients of $A_{n,k}$; $S_n^m[x]$ yields a number of known polynomials as special cases.

Throughout the paper we shall use the following notations:

$$P = m_2, n_2 \,;\, m_3, n_3 \quad , \quad Q = p_i^{(1)}, q_i^{(1)}; p_i^{(2)}, q_i^{(2)} : r$$

$$V = (a_j, \alpha_j)_{1,n_2}], (a_{ji}, \alpha_{ji})_{n_2+1,p_i^{(1)}}]; (c_j, \gamma_j)_{i,n_3}], (c_{ji}, \gamma_{ji})_{n_3+1,p_i^{(2)}}]$$

$$V' = (b_j, \beta_j)_{1,m_2}], (b_{ji}, \beta_{ji})_{m_2+1,q_i^{(1)}}]; (d_j, \delta_j)_{1,m_3}], (d_{ji}, \delta_{ji})_{m_3+1,q_i^{(2)}}]$$

and the result for binomial expansion

$$(x+a)^k = a^k \sum_{i=0}^{\infty} \binom{k}{i} (x/a) i \ , \quad |x/a| < 1 \quad \text{...(1.4)}$$

The formula given by Ross [5] as:

$$U_x^{\zeta,\upsilon}\{x^\lambda\} = \frac{\Gamma(\lambda+\zeta)}{\Gamma(\lambda+\zeta+\upsilon)}\, x^{-1}, \quad Re(\lambda) > -\zeta\,. \qquad \ldots(1.5)$$

2. Main Results. In this section the following two fractional integral formulae for the I-function of two variables are established as :

(I)

$$U_x^{\zeta,\upsilon}\left[x^\sigma(x+\alpha)^\rho(x+\beta)^\mu\, S_n^m[x^a(x+\alpha)^b(x+\beta)^c]\times I\begin{bmatrix} x^{h_1}(x+\alpha)^{k_1}(x+\beta)^{l_1}\ z_1 \\ x^{h_2}(x+\alpha)^{k_2}(x+\beta)^{l_2}\ z_2\end{bmatrix}\right]$$

$$= \alpha^\rho\beta^\mu \sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,\frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\,x^{\sigma+ak+t_1+t_2+ak}}{t_1!\,t_2!}\times$$

$$I^{m_1,n_1+3:P}_{p+3,q+3:Q}\left[\begin{matrix} x^{h_1}\alpha^{k_1}\beta^{l_1}\ z_1 \\ x^{h_2}\alpha^{k_2}\beta^{l_2}\ z_2\end{matrix}\,\middle|\,\begin{matrix} X : V \\ X' : V'\end{matrix}\right] \qquad \ldots(2.1)$$

where

$X = (1-\sigma-ak-t_1-t_2-\zeta : h_1,h_2),(-\rho-bk : k_1,k_2),(-\mu-ck : l_1,l_2),(e_p : E_p, E'_p)$

and

$X' = (f_q : F_q, F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1-t_2 : h_1,h_2),(-\rho-bk+t_1 : k_1,k_2),(-\mu-ck+t_2 : l_1,l_2)$

Provided that

(i) $Re(\upsilon) > 0$,

(ii) $Max\ \{|arg(x/\alpha)|,\ |arg(x/\beta)|\} < \pi$,

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1\min\limits_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right)+h_2\min\limits_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)+\zeta > 0.\right]$

(II)

$$U_x^{\zeta,\upsilon}\left[x^\sigma(x+\alpha)^\rho(x+\beta)^\mu\, S_n^m[x^a(x+\alpha)^b(x+\beta)^c]\times I\begin{bmatrix} x^{-h_1}(x+\alpha)^{-k_1}(x+\beta)^{-l_1}\ z_1 \\ x^{-h_2}(x+\alpha)^{-k_2}(x+\beta)^{-l_2}\ z_2\end{bmatrix}\right]$$

$$= \alpha^\rho\beta^\mu \sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,\frac{\alpha^{bk-t_1}\,\beta^{ck-t_2}\,x^{\sigma+ak+t_1+t_2}}{t_1!\,t_2!}\times$$

$$I^{m_1,n_1+3:P}_{p+3,q+3:Q}\left[\begin{matrix} x^{-h_1}\alpha^{-k_1}\beta^{-l_1}\ z_1 \\ x^{-h_2}\alpha^{-k_2}\beta^{-l_2}\ z_2\end{matrix}\,\middle|\,\begin{matrix} X : V \\ X' : V'\end{matrix}\right] \qquad \ldots(2.2)$$

where

$X = (e_p : E_p, E'_p),(\sigma+ak+t_1+t_2+\zeta+\upsilon : h_1,h_2),(1+\rho+bk-t_1 : k_1,k_2),(1+\mu+ck-t_2 : l_1,l_2)$,

and

$X' = (\sigma+ak+t_1+t_2+\zeta : h_1,h_2),(1+\rho+bk_1 : k_1,k_2),(1+\mu+ck : l_1,l_2),(f_q : F_q, F'_q)$,

provided that

(i) $Re(\upsilon) > 0$,

(ii) $Max\ \{|arg(x/\alpha)|,\ |arg(x/\beta)|\} < \pi$,

(iii) $Re\left[\sigma+ak+t_1+t_2-h_1 \underset{1\le j\le m_2}{Min}\left(\frac{b_j}{\beta_j}\right)-h_2 \underset{1\le j\le m_3}{Min}\left(\frac{d_j}{\delta_j}\right)\right]+\zeta > 0.$

Proof. To establish the fractional integral formula (2.1), first we express the general class of polynomials in the series form given by (1.3) and I-function of two variables in terms of Mellin-Barne's type contour integrals on the left hand side of (2.1).

Let

$$A = U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}(x+\beta)^{\mu}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\,x^{ak}(x+\alpha)^{bk}(x+\beta)^{ck}\times\frac{1}{(2\pi\omega)^2}\right.$$

$$\left.\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\,\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}x^{h_1\xi+h_2\eta}(x+\alpha)^{k_1\xi+k_2\eta}(x+\beta)^{I_1\xi+I_2\eta}\,d\xi d\eta\right] \quad ..(2.3)$$

Interchanging the order of Barne's type contour integrals and the fractional integral involved in the expression (2.3). Collecting the powers of x, $(x+\alpha)$, $(x+\beta)$ in the expression thus obtained and expanding $(x+\alpha)^{\rho}$ and $(x+\beta)^{\mu}$ using the binomial expansion (1.4), changing the order of Barne's type contour integrals and series, which is permissible under conditions stated, we get

$$A=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\Phi_1(\xi)\Phi_2(\eta)\Psi(\xi,\eta)Z_1^{\xi}Z_2^{\eta}\,\alpha^{\rho+bk+k_1\xi+k_2\eta}$$

$$\beta^{\mu+ck+I_1\xi+I_2\eta}\frac{\Gamma(\rho+bk+k_1\xi+k_2\eta+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta+1)}{t_1!\,t_2!\,(\rho+bk+k_1\xi+k_2\eta-t_1+1)\,\Gamma(\mu+ck+I_1\xi+I_2\eta-t_2+1)\,\alpha^{t_1}\alpha^{t_2}}\times$$

$$U_x^{\zeta,\upsilon}\left\{x^{\sigma+ck+t_1+t_2+h_1\xi+h_2\eta}\right\}d\xi d\eta \quad ...(2.4)$$

Now using the result (1.5), we arrive at the desired result (2.1). In the similar fashion we can easily establish the result (2.2).

3. Particular Cases

I. If we take $\alpha=\beta=1$ the result (2.1), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+1)^{\rho+\mu}S_n^m[x^a(x+1)^{b+c}]\times I\left[\begin{matrix}x^{h_1}(x+1)^{k_1+I_1}\,z_1\\ x^{h_2}(x+1)^{k_2+I_2}\,z_2\end{matrix}\right]\right]$$

$$=\sum_{t_1,t_2=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}\frac{x^{\sigma+ak+t_1}}{t_1!}I_{p+2,q+2:Q}^{m_1,n_1+2:P}\left[\begin{matrix}(x)^{h_1}z_1\\(x)^{h_2}z_2\end{matrix}\middle|\begin{matrix}Y:V\\Y':V'\end{matrix}\right], \quad ...(3.1)$$

where

$Y=(1-\sigma-ak-\zeta:h_1,h_2),(-\rho-\mu-bk-ck:k_1+I_1,k_2+I_2),(e_p:E_p,E'_p),$

$Y'=(f_q:F_q,F'_q),(1-\sigma-ak-\zeta-\upsilon-:h_1,h_2),(\rho-\mu-bk-ck+t_1:k_1+I_1,k_2+I_2),$

provided that

(i) $Re(\upsilon)>0$

(ii) $Max\ \{\ |arg\ (x/\alpha)|,\ |arg\ (x/\beta)|\} < \pi$

(iii) $Re\left[\sigma+ak+t_1+t_2+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right) + \zeta > 0\right]$.

(II) In (2.1) if we put $\mu = c = I_1 = I_2 = 0$, we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}(x+\alpha)^{\rho}\, S_n^m[x^a (x+\alpha)^b]\times I\begin{bmatrix} x^{h_1}(x+\alpha)^{k_1}\ z_1 \\ x^{h_2}(x+\alpha)^{k_2}\ z_2 \end{bmatrix}\right]$$

$$= \sum_{t_1=0}^{\infty}\ \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k}\, \frac{\alpha^{\rho+bk-t_1}\, x^{\sigma+ak+t_1}}{t_1!}\, I^{\,m_1,n_1+2\,:\,P}_{\,p+2,q+2\,:\,Q}\left[\begin{matrix} x^{h_1}\alpha^{k_1} z_1 \\ x^{h_2}\alpha^{k_2} z_2 \end{matrix}\ \middle|\ \begin{matrix} Y : V \\ Y' : V' \end{matrix}\right] \quad \text{... (3.2)}$$

where

$Y = (1-\sigma-ak-\zeta-t_1 : h_1,h_2),(-\rho-bk : k_1,k_2),(e_p : E_p, E'_p)$,

$Y' = (f_q : F_q, F'_q),(1-\sigma-ak-\zeta-\upsilon-t_1 : h_1,h_2),(-\rho-bk+t_1 : k_1,k_2)$,

provided that

(i) $Re\ (\upsilon) > 0$,

(ii) $Re\left[\sigma+ak+t_1+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right)\right] + \zeta > 0$

(iii) If we put $\rho = b = k_1 = k_2 = 0$ in (3.2), we get

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\, S_n^m[x^a]\ I\begin{bmatrix} x^{h_1}\ z_1 \\ x^{h_2}\ z_2 \end{bmatrix}\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^{\sigma+ak}\, I^{\,m_1,n_1+1\,:\,P}_{\,p+1,q+1\,:\,Q}\left[\begin{matrix} x^{h_1} z_1 \\ x^{h_2} z_2 \end{matrix}\ \middle|\ \begin{matrix} Y : V \\ Y' : V' \end{matrix}\right]$$

where ... (3.3)

$Y = (1-\sigma-ak-\zeta : h_1,h_2)(e_p : E_p, E'_p)$

$Y' = (f_q : F_q, F'_q),(1-\sigma-ak-\zeta-\upsilon : h_1,h_2)$

Provided that

(i) $Re\ (\upsilon) > 0$,

(ii) $\left[Re\ \ \sigma+ak+h_1 \min_{1\le j\le m_2}\left(\frac{b_j}{\beta_j}\right) + h_2 \min_{1\le j\le m_3}\left(\frac{d_j}{\delta_j}\right) + \zeta > 0\right]$

(iv) In (3.3), if we put $m_1 = n_1 = p = q = n_3 = p_i^{(2)} = 0,\ m_3 = q_i^{(2)} = 1$ and $z_2 \to 0$ the result converts in the form of Saxena's [6] I-function of one variable

$$U_x^{\zeta,\upsilon}\left[x^{\sigma}\, S_n^m[x^a]\ I^{\,m_2,n_2}_{\,p_i^{(1)},q_i^{(1)}\,:\,r}[x^{h_1}, z_1]\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^{\sigma+ak}$$

$$I^{\,m_2,n_2+1}_{\,p_i^{(1)}+1,q_i^{(1)}+1\,:\,r}\left[x^{h_1} z_1\ \middle|\ \begin{matrix} (1-\sigma-ak-\zeta : h_1),V \\ V',(1-\sigma-ak-\zeta-\upsilon : h_1) \end{matrix}\right] \quad \text{...(3.4)}$$

Provided that

(i) $Re\,(\upsilon) > 0,$

(ii) $Re\left[\sigma + ak + h_1 \min_{1 \le j \le m_1} \left(\frac{b_j}{\beta_j}\right)\right] + \zeta > 0.$

(V) In (3.4) if we take $m=1$, $A_{n,k} = \binom{n+\gamma}{n} \frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k}$ then the general class of polynomials $S_n^m[x^a]$ reduces to Jacobi polynomials, i.e., $S_n^1[x^a] \to P_n^{(\gamma,\delta)}[1-2.x^a]$

$$U_x^{\zeta,\upsilon}\left[x^{\sigma} P_x^{(\gamma,\delta)}[1-2x^a]\, I_{p_i^{(1)},q_i^{(1)}:r}^{m_2,n_2}\,[x^{h_1}, z_1]\right] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} \binom{n+\gamma}{n}$$

$$\frac{(\gamma+\delta+n+1)_k}{(\gamma+1)_k} x^{\sigma+ak}\; I_{p_i^{(1)}+1,q_i^{(1)}+1:r}^{m_2,n_2+1}\left[x^{h_1} z_1 \middle| \begin{matrix} (1-\sigma-ak-\zeta : h_1), V \\ V', (1-\sigma-ak-\zeta-\upsilon : h_1) \end{matrix}\right] \qquad (3.5)$$

Convergence conditions are same as in result (3.4).

In the similar manner we can establish the particular cases for the result (2.2).

REFERENCES

[1] A. Erdeilyi, On fractional integration and its applications to the theory of Hankel transforms. *Quart. J. Math. (Oxford)* **11** (1940), 293.

[2] Anil Goyal and R.D. Agrawal. Integral involving the product of *I*-function of two Variables, *Journal M.A.C.T.*, **28** (1995) 147-155.

[3] Shailesh Jaloree, Anil Goyal and R.D. Agrawal, Fractional integral formulae involving the product of Hypergeometric function, A general class of polynomials & *I*-Function of two variables-I, (Under Communication).

[4] H. Kober, On fractional integrals and derivatives., *Quart.J. Math. (Oxford)* **11** (1940), 193.

[5] B. Ross. *Fractional Calculus and its Applications Lecture Notes in Math.* **457** (1975), Springer-Verlag, NewYork.

[6] V.P. Saxena, Formal solution of certain new pair of dual integral equations involving *H*-functions. *Proc. Nat. Acad. Sci., India* **52** (A) (1982) 366-375.

[7] H.M. Srivastava, K.C. Gupta and S.P. Goyal *The H-Fuctions of One and Two Variables with Applications*, South Asian Publishers, New Delhi, (1982)

[8] H.M. Srivastava, A contour integral involving Fox's *H*-function, *Indian J. Math.*, **14** (1972), 1-6.

Jñānābha, Vol. 30, 2000

ON RELATIVELY SINGULAR MAPS

By

Anjali Srivastava and Abha Khadke
School of Studies in Mathematics
Vikram University, Ujjain-456010, M.P.

(Received : November 20, 1999)

ABSTRACT

In this paper we introduce the concept of relatively singular maps between locally compact spaces and obtain various results concerned to singular maps in this new setting.

1. Introduction. Throughout the paper a space will mean a locally compact Hausdorff space and a map will mean a continuous map.

The notion of the singular set of a mapping was defined and investigated by G.T. Whyburn [14] and G.L. Cain [2,3]. Later the idea became quite useful to obtain a compactification of a locally compact space X with a given compact space K as a remainder. The compactification obtained by this method is called a *singular compactification*. This laid a new step in the theory of compactifications, one of the most active areas of researches in general topology. A few celebrated names in this area are Banaschewski, Cain, Chandler, Conway, Faulkner, Magill Jr., Walker and Whyburn [2,3,4,5,6,7,8,12, 13,14].

A point y in Y is called a *singular point* of $f : X \to Y$ if the closure $cl\, f^{-1}(V)$ of $f^{-1}(V)$ is not compact for any neighbourhood V of y. The set of singular points of f is denoted by $S(f)$. Clearly it is a closed set of Y. If for a map $f{:}X\to Y$, $S(f) = Y$, f is called a *singular map*.

Let $f : X\to Y$ and $\alpha{:}Y\to Y$ be a self map. A point $y \,\varepsilon Y$ is called a *relatively singular point* of the map $f(w.r.t.\alpha)$ if the closure $cl\,(\alpha\, o\, f)^{-1}(U)$ of $(\alpha o f)^{-1}(U)$ is not compact for any neighbourhood U of y. If every point of Y is relatively singular, we call f to be *relatively singular map* ($w.r.t.\alpha$ $:Y\to Y$).

The singular map and relatively singular map may differ but when α is equal to identity map on Y, both concepts coincide. A singular map is always relatively singular but the converse is not true.

1.1 Example. The indentity map $I{:}R^2 \to R^2$ is not singular but I is relatively singular with respect to the first projection $\pi_1 : R^2 \to R$.

In section *2* we show that the product of a relatively singular map

with any map is relatively singular. Also the induced map of a relatively singular map $f:X \to Y$ on respective cones remains relatively singular. Moreover an analogue of Pasting lemma is obtained for relatively singular maps.

A G-space is a Hausdorff space X on which a topological group G acts continuously. The orbit space of a G-space is denoted by X/G and the map which assigns to x in X its orbit is the orbit map on X. An equivariant map f from a G-space X to a G-space Y induces a continuous map f_G on orbit spaces which sends the orbit of x in X to the orbit of $f(x)$. Let H be a compact subgroup of G and let X be an H-space. Then for $h \varepsilon H$ and $(g,x) \varepsilon G \times X$, $h(g,x) = (gh^{-1}, h.x)$ defines an action of H on $G \times X$; the orbit space $G \times_H X$ of the H-space $G \times X$ is called the *twisted product* of G and X. For H-spaces X and Y, an equivariant map $f : X \to Y$ determines a continuous map $f_H : G \times_H X \to G \times_H Y$, which sends $[g,x]$ in $G \times_H X$ to $[g,f(x)]$ in $G \times_H X$ [1].

A map $f : X \to Y$ will be called *compact* if the inverse image of a compact set of Y is a compact set of X.

We consider G-spaces and equivariant maps in section 3 and obtain that an equivariant relatively singular map from a G-space X to a G-space Y induces a relatively singular map on orbit spaces.

2. Relatively Singular maps and Various Structures

In this section we obtain some results for relatively singular maps which are already established for singular maps [10].

2.1 Proposition. Let $f : X \to Y$, $h : X' \to Y'$ and $\alpha : Y \to Y$ be maps, If f is relatively singular (*w.r.t.* α), then $f \times h : X \times X' \to Y \times Y'$ defined by

$$(f \times h)(x, x') = \{ f(x), h(x') \}$$

is relatively singular (*w.r.t.* $\alpha \times I$) where $I : Y' \to Y'$ is the idnetity map.

Proof. Suppose that $f \times h : X \times X' \to Y \times Y'$ is not relatively singular (*w.r.t.* $\alpha x I$). Then there exists a basic open set $U \times U'$ of $Y \times Y'$ such that $cl[(\alpha \times I) \circ (f \times h)]^{-1} (U \times U')$ is compact.

Since

$$cl[(\alpha \times I) \circ (f \times h)]^{-1} (U \times U')$$
$$= cl\,(\alpha o f)^{-1} (U) \times cl\,(I \times h)^{-1} (U')$$
$$\Rightarrow cl\,(\alpha \circ f)^{-1} (U) \text{ is compact.}$$

A contradiction to the hypothesis.

The following proposition gives an analogue of Pasting Lemma for relatively singular maps.

2.2 Proposition. Let X, Y be locally compact spaces and A,B be

closed set of X such that $A \cup B = X$. Let $f:A \to Y$ and $g:B \to Y$ be two relatively singular maps $(w.r.t.\alpha: Y \to Y)$ such that $f(x) = g(x)$ for $x \,\varepsilon\, A \cap B$. Then the map $h:X \to Y$ defined by

$$h(x) = \begin{cases} f(x) & \text{if } x \,\varepsilon\, A \\ g(x) & \text{if } x \,\varepsilon\, B \end{cases}$$

is relatively singular $(w.r.t.\alpha: Y \to Y)$.

Proof. Let U be an open set of Y. Then

$(\alpha o h)^{-1}(U) = (\alpha \, o \, f)^{-1}(U) \cup (\alpha o g)^{-1}(U) \Rightarrow cl\,(\alpha o h)^{-1}(U) = cl(\alpha \, o \, f)^{-1}(U) \cup cl\,(\alpha o g)^{-1}(U)$ ..

Since f and g are relatively singular maps, $cl\,(\alpha \, o \, f)^{-1}(U)$ and $cl\,(\alpha o g)^{-1}(U)$ both are noncompact.

$\Rightarrow (\alpha \, o \, h)^{-1}(U)$ is non-compact.

2.3 Definition. Let X be a topological space, the *Cone* T_X over X is the quotient space $(X \times I)/A$, where A is the subset of $X \times I$ given by $X \times \{1\}$.

2.4 Proposition. Let $f: X \to Y$ be a reltively singular map $(w.r.t.\alpha: Y \to Y)$. If the quotient maps p,q are compact maps, then the induced map $Tf: Tx \to Ty$ is relatively singular $(w.r.t.T\alpha: Ty \to Ty)$.

Proof Consider the following commutative diagram.

$$\begin{tikzcd} X \times I \arrow[r, "(\alpha \times I)\, o\, (f \times I)"] \arrow[d, "p"'] & Y \times I \arrow[d, "q"] \\ Tx \arrow[r, "T\alpha \, o \, Tf"'] & Ty \end{tikzcd}$$

If $f: X \to Y$ be relatively singular $w.r.t.\alpha: Y \to Y$, then $f \times I : X \times I \; Y \times I$ be relatively singular $w.r.t.\alpha \times I: Y \times I \to Y \times I$.

Suppose $Tf: Tx \to Ty$ is not relatively singular, then there is an open set U of Ty satisfying that $cl\,(T\alpha o Tf)^{-1}(U)$ is compact.

$\Rightarrow p^{-1}(cl(T\alpha o\, Tf)^{-1}(U))$ is compact. From continuity of q and commutativity of the above diagram, it follows that

$$\begin{aligned} p^{-1}(cl(T\alpha o\, Tf)^{-1}(U)) &\supseteq cl\,(p^{-1}(T\alpha o\, Tf)^{-1}(U)) \\ &= cl\,(p^{-1}(Tf^{-1}(T\alpha^{-1}(U)))) \\ &= cl\,[(T\alpha \, o \, Tf)\, o\, (p)]^{-1}(U) \\ &= cl\,[q o \{(\alpha \times I) \text{ o } (f \times I)\}]^{-1}(U) \\ &= cl\,[(\alpha \times I) \text{ o } (f \times I)]^{-1} q^{-1}(U) \end{aligned}$$

Thus $cl\,[(\alpha \times I) \text{ o } (f \times I)]^{-1} q^{-1}(U)$ is compact, a contradiction to the hypothesis that $f \times I$ is relatively singular.

3. Relatively Singular Maps and G-spaces. In this section X and Y will denote G-spaces, where G is a compact topological group and the spaces X and Y are respectively locally compact and compact.

3.1 Proposition. Let $f:X\to Y$ be an equivariant and relatively singular map ($w.r.t. \alpha: Y\to Y$). Then the induced map $f_G: X/G\to Y/G$ is relatively singular ($w.r.t. \alpha_G: Y/G\to Y/G$).

Proof. Consider the following commutative diagram.

$$\begin{array}{ccc} X & \xrightarrow{\alpha\, o\, f} & Y \\ q_x\downarrow & & \downarrow q_y \\ X/G & \xrightarrow[\alpha_G\, o\, f_G]{} & Y/G \end{array}$$

where q_x and q_y are orbit maps.

Suppose that f_G is not relatively singular. Then there is a point $G(y)$ in Y and an open set U of Y/G containing $G(y)$ such that $cl(\alpha_G o f_G)^{-1}(U)$ is compact. Since q_x is a compact map,

$q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U))$ is compact. From continuity of q_x and commutativity of the above diagram, it follows that

$$\begin{aligned} q_x^{-1}(cl(\alpha_G o f_G)^{-1}(U)) &\supseteq cl\,(q_x^{-1}(\alpha_G o f_G)^{-1}(U)) \\ &= cl\,(q_x^{-1}(f_G^{-1}(\alpha_G^{-1}(U)))) \\ &= cl\,((\alpha_G o f_G)\, o q_x)^{-1}(U) \\ &= cl\,(q_y\, o\, (\alpha o f))^{-1}(U) \\ &= cl\,(f^{-1}(\alpha^{-1}(q_y^{-1}(U)))). \end{aligned}$$

Thus $cl(f^{-1}(\alpha^{-1}(q_y^{-1}(U))))$ is compact, a contradication to the hypothesis that f is relatively singular.

3.2. Proposition. Let $f:X\to Y$ be an equivariant and relatively singular map ($w.r.t. \alpha: Y\to Y$). Then the induced map $f_H: G\times_H X\to G\times_H Y$ is relatively singular $w.r.t. \alpha_H: G\times_H Y\to G\times_H Y$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} G\times X & \xrightarrow{(I_G\times\alpha)\, o\, (I_G\times f)} & G\times Y \\ q_x\downarrow & & \downarrow q_y \\ G\times_H X & \xrightarrow[\alpha_H\, o\, f_H]{} & G\times_H Y \end{array}$$

Suppose that f_H is not relatively singular. Then there is an open set U of $G\times_H Y$ such that $cl\ (\alpha_H o f_H)^{-1}(U))$ is compact, then $q_x^{-1}\,(cl\ (\alpha_H\ o\ f_H)^{-1}(U))$ is also compact. From continuity of q_x and commutativity of the above diagram it follows that $q_x^{-1}(cl(\alpha_H o f_H)^{-1}(U)) \supseteq cl\,(q_x^{-1}\,(\alpha_H o f_H)^{-1}(U))$

$$= cl\,(q_x^{-1}(f_H^{-1}(\alpha_H^{-1}(U))))$$
$$= cl\,((\alpha_H o f_H)\, o q_x)^{-1}(U)$$
$$= cl\,(q_y\, o\, (((I_G \times \alpha)\, o\, (I_G \times f))^{-1}(U)$$
$$= cl\,[((I_G \times \alpha)\, o\, (I_G \times f))^{-1}\, q_y^{-1}\,(U))]$$

is compact, a contradiction to the hypothesis that $I_G \times f$ is relatively singular.

REFERENCES

[1] G.E. Bredon, *Introduction to Compact transformation groups*, Academic Press, New York, London.

[2] G.L. Cain Jr., Compact and related mappings, *Duke Math. Jour* **33** (1966), 639-645.

[3] G.L. Cain Jr., Mappings with prescribed singular sets, *Neiuw Arch. Wisk*, **17** (1969), 200-203.

[4] G.L. Cain Jr., R.E. Chandler and Gary D. Faulkner, Singular sets and remainders, *Trans. Amer. Nath. Soc.* **268** (1981), 161-171.

[5] R.E. Chandler and Gary D. Faulkner, Singular Compactifications: The order structure, *Proc. Amer. Math. Soc.* **100** (1987), 377-382.

[6] John B. Conway, Projections and retractions, *Proc. Amer. Math. Soc.* **1** (1966), 843-847.

[7] Gary D. Faulkner, Compactifications whose remainders are retracts, *Proc. Amer. Math. Soc.* **103** (1988), 984-988.

[8] K.D. Magill Jr., More on remainders of spaces in compactifications, *Bull. Acad. Polon. Sci Ser. Sci Math. Astronom. Phys.* 18 (1970), 449-451.

[9] J.R. Munkres, *Topology : A first Course*, Prentice-Hall of India Private Limited, New Delhi (1983).

[10] Anjali Srivastava and Abha Khadke, On singular and H-singular maps, *Bull. Cal. Math Soc.* **90**, 331-336 (1998).

[11] Kavita Srivastava and Anjali Srivastava, A survey on singular compactifications, *J. Ramanujan Math. Soc.* **9** (1994), 109-140.

[12] R.C. Walker, *The Stone-Cech Compactification*, Springer-Verlag, Berlin, Heidelberg, New York (1974).

[13] G.T. Whyburn, A unified space for mappings, *Trans. Amer. Math. Soc.* **74** (1953), 344-350.

[14] G.T. Whyburn, Compactification of mappings, *Math. Ann.* **166** (1966), 168-174.

Jñānābha, Vol. 30, 2000

APPROXIMATING COMMON FIXED POINTS OF TWO NONEXPANSIVE MAPPINGS IN BANACH SPACES

By

Anil Rajput and Abha Gupta

Department of Mathematics,

Sadhu Vaswani College, Sant Hirdram Nagar, Bhopal-462003, M.P.

(Received : December 20, 1999)

ABSTRACT

Let C be a nonempty subset closed convex subset of a real Banach space R and let U, T be nonexpansive mappings of C into itself. In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of mappings U and T:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{1}{n^2}\alpha_n \sum_{i,j=0}^{n-1} U^i T^j \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in [0,1]. Using some ideas in the nonlinear ergodic theory, we prove that the iterates converge weakly to a common fixed point of the nonexpansive mappings T and U in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

1. Introduction. Let C be a nonempty closed convex subset of a Banach space R. Then, a mapping $T: C \to C$ is called nonexpansive if

$$\| Tx - Ty \| \leq \|x - y\| \qquad \text{for all } x,y \in C.$$

We denote by $F(T)$ the set of fixed points of T. Mann [5] introduced an iteration procedure for approximating fixed points of a mapping T in a Hilbert space as follows:

where $\{\alpha_n\}$ is a sequence in [0,1]. Later, Reich [7] discussed this iteration procedure in a uniformly convex Banach space whose norm is Frechet differentiable.

In this paper, we consider the following iteration procedure of Mann's type for approximating common fixed points of two nonexpansive mappings in a Banach space:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \frac{\alpha_n}{n^2} \sum_{i,j=0}^{n-1} U^i T^j x_n \text{ for every } n \geq 1,$$

where $\{\alpha_n\}$ is a sequence in [0,1] and U, T are nonexpansive mappings of C into itself. Using some ideas in the nonlinear ergodic theory and an inequality obtained by *Xu [9]*, we prove that the iterates converge weakly to a common fixed point of the two nonexpensive mappings in a uniformly

convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable.

Through this paper, we assume that R is a real Banach space. We denote by R^* the dual space of R and also denote (y,x^*) the value of $x^* \in R^*$ at $y \in R$. We write $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) to indicate that the sequence $\{X_n\}$ of vectors converges weakly to x. Similarly $x_n \to x$ (or $\lim_{n\to\infty} x_n = x$) will symbolize strong convergence. We denote by N the set of all positive integers. For a subset A of R, coA and $\overline{coA}$ mean the convex hull of A and the closure of the convex hull of A, respectively. We say that R satisfies Opial's condition [6] if for any sequence $\{X_n\} \subset R$ with $x_n \to x \in R$, the inequality

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$$

holds for every $y \in R$ with $y \neq x$. It is known that all Hilbert spaces and l^p with $1 < p < \infty$ satisfy Opial's condition. It is also known that every separable Banach space can be equivalently renormed so that it satisfies Opial's condition [3]. We also know that if the Banach space R has a duality mappings which is weakly sequentially continuous at 0, then R satisfies Opial's condition [4]. However, the space L^p with $1 < p < \infty$ and $p \neq 2$ do not satisfy Opial's condition [6]. Let R be a Banach space.

Then, the norm of R is said to be Gatcaux differentiable if

$$\lim_{t\to 0} (\| x + ty \| - \| x \|)/t$$

exists for each x and y in U_R, where $U_R = \{x \in R : \| x \| = 1\}$. It is said to be Frechet differentiable if for each x in U_R, this limit is attained uniformly for y in U_R. Let C be a closed convex subset of R and let T be a mapping of C into itself. Then, for each $\varepsilon > 0$, we define the set $F_\varepsilon(T)$ to be

$$F_\varepsilon(T) = \{x \in C : \| Tx - x \| \leq \varepsilon\}$$

The following lemmas were proved in [2].

Lemma1. Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then, for each $\in > 0$, there exits $\delta > 0$ such that

$$co\, F_\delta(T) \subset F_\varepsilon(T)$$

for every nonexpansive mappings T of C into itself.

Lemma.2 Let R be a uniformly convex Banach space and let C be a nonempty bounded closed convex subset of R. Then,

$$\lim_{n\to\infty} \sup_{\substack{x \in C \\ T \in N(C)}} \|1/(n+1) \sum_{i=0}^{n} T^i x - T(1/(n+1) \sum_{i=0}^{n} T^i x)\| = 0,$$

where $N(C)$ denotes the set of all nonexpansive mappings of C into itself.

Lemma.3. Let C be a nonempty closed convex subset of a Banach space R. Let U and T be a nonexpansive mappings of C into itself such that $UT=TU$ and $F(S) \cap F(T) \neq \phi$. Now consider the following iteration scheme:

$$x_1 = x \in C,\ x_{n+1} = (1-\alpha_n)\, x_n + \alpha_n\ 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x_n \qquad \text{...(1)}$$

for every $n \in N$,

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Then, for any $n \in N$ putting

$$T_n\, x = (1-\alpha_n)\, x + \alpha_n\ 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x \qquad \text{for every } n \in C,$$

the mapping T_n of C into itself is also nonexpansive. In fact, let $x,y \in C$. Then, we obtain

$$\begin{aligned}
\| 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y \| &\leq 1/n^2 \sum_{i,j=0}^{n-1} \| U^i\, T^j x - U^i\, T^j y \| \\
&\leq 1/n^2 \sum_{i,j=0}^{n-1} \| x - y \| \\
&= \| x - y \|.
\end{aligned}$$

$$\begin{aligned}
\|T_n x - T_n y\| &= \|\{(1-\alpha_n)\, x + \alpha_n\ 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x\} - \\
&\quad \{(1-\alpha_n)\, y + \alpha_n\ 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y\}] \| \\
&\leq (1-\alpha_n)\ \|x-y\| + \alpha_n\ \|1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j x - \\
&\qquad 1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j y \| \\
&\leq (1-\alpha_n)\, \| x-y \| + \alpha_n\, \| x-y \| \\
&= \| x-y \|.
\end{aligned}$$

Further, we have $F(T) \cap F(U) \subset F(1/n^2 \sum_{i,j=0}^{n-1} U^i\, T^j) \subset F(T_n)$ for every $n \in N$ and hence $F(T) \cap F(U) \subset \bigcap_{n=1}^{\infty} F(T_n)$.

The iterates $\{x_n\}$ defined by (1) can be written as

$$x_{n+1} = T_n,\ T_{n-1} \ldots T_1 x_1,. \qquad \text{... (2)}$$

Putting

$U_n = T_n\, T_{n-1} \ldots T_1 x_{n+1}$, we get

$$x_{n+1} = U_n\, x_1 \qquad \text{... (3)}$$

Using Lemmas 1 and 2, we can prove the follwing lemma :

Lemma 3. Let C be a nonempty bounded closed convex Banach space R. Let U and T be nonexpansive mapping of C into itself with $UT = TU$. Then,

$$\lim_{n\to\infty} sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

and

$$\lim_{n\to\infty} sup_{x\in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j y) \| = 0,$$

Proof. Let $\varepsilon > 0$. From Lemma 1, we know that there exits $\delta > 0$ such that

$$\overline{co}\, F_\delta (S) \subset F_\varepsilon (S) \qquad \dots (4)$$

for every nonexpansive mapping S of C into itself. From Lemma 2, we also have

$$\lim_{n\to\infty} sup_{y\in C} \|1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y) \| = 0.$$

Then, there exits $n_1 \in N$ such that

$$sup_{y\in C} \| 1/n \sum_{i=0}^{n-1} U^i y - U(1/n \sum_{i=0}^{n-1} U^i y) \| < \delta,$$

for every $n \geq n_1$. Then, we obtain that

$$1/n \sum_{i=0}^{n-1} U^i y \in F_\delta (U) \subset co\, F_\delta (U) \qquad \dots (5)$$

for every $y \in C$ and $n \geq n_1$. Let $1,p \in N$. Then, we have, for any $n \in N$ with $n>1,p$ and $x \in C$,

$$\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \|$$

$$\leq \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - (1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$+ \| U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+l} x - U(1/n^2 \sum_{i,j=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

$$\leq 2 \| 1/n^2 \sum_{i=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+l} x) \|$$

$$+ \| 1/n^2 \sum_{i=0}^{n-1} U^{i+l} T^{j+p} x - U(1/n^2 \sum_{i=0}^{n-1} U^{i+p} T^{j+p} x) \|$$

and

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \|$$

$$= \| 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} U^i T^j x + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} U^i T^j x - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1} U^i T^j x - \sum_{i=1}^{n-1+l} \sum_{j=n}^{p+n-l} U^i T^j x) \|$$

$$\leq 1/n^2 (\sum_{i=0}^{n-1} \sum_{j=0}^{p-1} \| U^i T^j x \| + \sum_{i=0}^{l-1} \sum_{j=0}^{n-1} \| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{n-1}$$

$$\| U^i T^j x \| - \sum_{i=n}^{n+l-1} \sum_{j=p}^{p+n-l} \| U^i T^j x \|)$$
$$\leq 1/n^2 \{ np + l(n-p) + l(n+p) + np \} M$$
$$\leq (2M(l+p)/n),$$

where $M = sup_{z \in C} \| z \|$. Then, there exits $n_0 \in N$ such that $n_0 > max\{n1, l, p\}$ and $(2M(l+p)/n) < \varepsilon$ for every $n \geq n_0$. This is implies that

$$I = \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - 1/n^2 \sum_{i,j=0}^{n-1} U^{i+l} T^{j+p} x) \| < \in$$

for every $x \in C$. Next, we prove that

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co}\, F_\delta (U).$$

for every $x \in C$, $m,q \in N$ and $n \geq n_1$. If not, we have that for some $m,q \in N$, $n \geq n_1$ and $x \in C$

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \notin \overline{co}\, F_\delta (U).$$

From the separation theorem, there exits $y^*_1 \in R^*$ such that

$$(1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x y^*_1) < inf \{(z, y^*_1) : z \in \overline{co}\, F_\delta (U)\}. \quad ...(6)$$

Then from (5) we obtain

$$inf \{(z,y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq inf \{(1/n \sum_{i=0}^{n-1} U^i x\, y^*_1) : x \in C\}$$
$$\leq (1/n \sum_{i=0}^{n-1} U^i y, y^*_1)$$

for all $y \in C$ and $n \geq n_1$. Then we have that for any $j \in \{0,1,2,3, ...,n-1\}$

$$inf \{(z,y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x), y^*_1)$$

and hence

$$inf \{(z,y^*_1) : z \in co\, F_\delta (U)\} \leq 1/n \sum_{i=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$

Therefore, From (6),

$$inf \{(z,y^*_1) : z \in \overline{co}\, F_\delta (U)\} \leq 1/n \sum_{j=0}^{n-1} (1/n \sum_{i=0}^{n-1} U^i (U^m T^{j+q} x, y^*_1).$$
$$< inf \{(z,y^*_1) : z \in \overline{co}\, F_\delta (U)\}.$$

This is contraction. Hence, from (4), we have

$$1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x \in \overline{co}\, F_\delta (U) \subset F_\varepsilon (U) \quad ...(7)$$

for every $m,q \in N$, $x \in C$ and $n \geq n_1$ then, it follows from (7) that

$$sup_{\substack{x \in C \\ m,n \in N}} \| 1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x - U\ (1/n^2 \sum_{i,j=0}^{n-1} U^{i+m} T^{j+q} x) \| < \varepsilon$$

for every $n \geq n_1$. Hence, we obtain

$$\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \ (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| < 2\varepsilon + \varepsilon = 3\varepsilon$$

for every $x \in C$ and $n \geq n_0$. Therefore,

$$\lim_{n\to\infty} sup_{x \in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - U \ (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| = 0.$$

Similary, we have

$$\lim_{n\to\infty} sup_{x \in C} \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x - T (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x) \| = 0.$$

Lemma 4. Let R be a Banach space and let C be a nonempty closed convex subset of R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. Let we a common fixed point of T and U. Then, $lim \ \| x_n - \omega \|$ exists.

Proof. Let ω be a common fixed point of T and U. Then, we have

$$\begin{aligned}
\| x_{n+1} - \omega \| &= \| (1-\alpha_n)x_n + \alpha_n 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \| \\
&\leq (1-\alpha_n) \| x_n - \omega \| + \alpha_n \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \| \\
&\leq (1-\alpha_n) \| x_n - \omega \| + \alpha_n \| x_n - \omega \| \\
&= \| x_n - \omega \|
\end{aligned}$$

and hence $\lim_{n\to\infty} \| x_n - \omega \|$ exists.

Lemma 5. Let C be a nonempty closed convex subset of a uniformly convex Banach space R. Let U and T be nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$.

Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $0 \leq \alpha_n \leq a$ for some a with $0 < a < 1$. Then,

$$\lim_{n\to\infty} \| Tx_n - x_n \| = \lim_{n\to\infty} (Ux_n - x_n \| = 0.$$

In particular, $x_n \to y_0$ implies $y_0 \in F(T) \cap F(U)$.

Proof. For $x \in C$ and $f \in F(U) \cap F(T)$, put $r = \| x - f \|$ and set $X = \{u \in R: \| u - f \| \leq r\} \cap C$. then, X is a nonempty bounded closed convex subset of C which is T, U-invariant and contains $x_1 = x$. So, without loss of generality, we may assume that C is bounded. Let w be a common

fixed point of T and U. Then, there exits a continuous, strictly increasing and convex function $g : [0,\infty)\ [0,\infty)$
such that $g(0) = 0$ and

$$\| x_{n+1} - \omega \|^2 = \| (1-\alpha_n)(x_n - \omega) + \alpha_n (1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2$$

$$- \alpha_n (1-\alpha_n) g(\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

for all $n \in N$. Then, since $\alpha_n \leq a$, we have

$$\alpha_n (1-a) g(\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|$$

$$\leq \alpha_n (1-\alpha_n) g(\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|)$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n A | 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - \omega \|^2 - \| (x_{n+1} - \omega \|^2$$

$$\leq (1-\alpha_n) \| x_n - \omega \|^2 + \alpha_n \| x_n - \omega \|^2 - \| (x_{n+1} - \omega \|^2$$

$$\leq | |(x_n - \omega) \|^2 - \| (x_{n+1} - \omega \|^2.$$

So, from Lemma 4, we obtain

$$\lim_{n\to\infty} (1-\alpha_n) g(\| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \|) = 0.$$

Since g is continuous, strictly increasing and stisfies $g(0) = 0$, we have

$$\lim_{n\to\infty} (1-\alpha_n) \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_n \| = 0. \qquad \dots (8)$$

It follows from the definition of $\{x_n\}$ that

$$x_{n+1} - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n = (1-\alpha_n)(x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n).$$

Since

$$\| Tx_{n+1} - x_{n+1} \| \leq \| Tx_{n+1} - T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n) \|$$

$$+ \| T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \|$$

$$+ \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \|$$

$$\leq 2 \| 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n - x_{n+1} \| + \| T(1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n)$$

$$- 1/n^2 \sum_{i,j=0}^{n-1} U^i T^j x_n \|$$

from (8) and Lemma 3, we have

$$\lim_{n\to\infty} \| Tx_n - x_n \| = 0.$$

Similary, we have

$$\lim_{n\to\infty} \| Ux_n - x_n \| = 0.$$

Assume $x_n \to y_n$. Then, since I-T and I-U are semiclosed by [1], we obtain that y_0 is a common fixed point of T and U.

3. Main Result

Theorem 1. Let R be a uniformly convex Banach space which satisfies Opial's condition or whose norm is Frechet differentiable. Let C be a nonempty closed convex subset of R. Let U and T be a nonexpansive mappings of C into itself such that $UT = TU$ and $F(U) \cap F(T) \neq \phi$.

Suppose $x_1 = x \in C$ *and* $\{x_n\}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j x_n \qquad \text{for every } n \in N,$$

where $\{\alpha_n\}$ is a sequence in [0,1]. If $\{\alpha_n\}$ is chosen so that $\alpha_n \in [0,a]$ for some a with $0 < a < 1$, then $\{x_n\}$ converges weakly to a common fixed point z_0 of T and U.

Proof:. Let $x \in C$. We first assume that R satisfies Opial's condition. Let ω be a common fixed point of T and U. Then, from lemma 4, $\lim_{n\to\infty} \| x_n - \omega \|$ exits. As in the proof of Lemma 5, we may assume that C is bounded. Since R is reflexive, $\{x_n\}$ must contain a subsequence which converges weakly to a point in C. So, let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be a two subsequence of $\{x_n\}$ such that $x_{n_i} \to z_1$ and $x_{n_j} \to z_2$. Then, from Lemma 5, we have that z_1 and z_2 are common fixed points of T and U. Next, we show $z_1 = z_2$. If not, from Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \| x_n - z_1 \| &= \lim_{i\to\infty} \| x_{n_i} - z_1 \| \\
&< \lim_{i\to\infty} \| x_{n_j} - z_2 \| = \lim_{n\to\infty} \| x_n - z_2 \| \\
&\qquad = \lim_{j\to\infty} \| x_{n_j} - z_2 \| \\
&< \lim_{j\to\infty} \| x_{n_j} - z_1 \| - \lim_{n\to\infty} \| x_n - z_1 \|
\end{aligned}$$

This is contradiction. Hence, we obtain $x_n \to y_0 \in F(T) \cap F(U)$.

Next, we assume that R has a Frechet differentiable norm. As in the proof of Lemma 5, we may assume that C is bounded. So, there exists a subsequence $\{x_n\}$ of $\{x_n\}$ such that $x_{n_i} \to y_0$. Then, from Lemma 5, we obtain $y_0 \in F(T) \cap F(U)$. Putting

$$T_n y = (1-\alpha_n)y + \alpha_n(1/n^2) \sum_{i,j=0}^{n-1} U^i T^j y$$

and

$U_n y = T_n T_{n-1} T_{n-2} \ldots T_1 y$ for all $n \in N$ and $y \in$ C, form (3), we have

2. The Fractional Integral Operators. We recall here a few definitions and properties of operators used in solving the triple and quadruple integral equations. Lowndes [4] has defined the following operators:

(4) $$I_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma\alpha} \int_a^b (x^\sigma - t^\sigma)^{\alpha-1} t^{\sigma(\eta+1)-1} f(t)\, dt, \ \alpha > 0,$$

(5) $$= \frac{x^{1-\sigma(\alpha+\eta+1)}}{\Gamma(1+\alpha)} d/dx \int_a^b (x^\sigma - t^\sigma)^{\alpha} t^{\sigma(\eta+1)-1} f(t)\, dt, \ -1 < \alpha < 0,$$

(6) $$K_{\eta,\alpha}(c, d;\sigma) f(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma\alpha} \int_c^d (t^\sigma - x^\sigma)^{\alpha-1} t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt, \ \alpha > 0,$$

(7) $$= \frac{x^{\sigma(\eta-1)+1}}{\Gamma(1+\alpha)} d/dx \int_c^d (t^\sigma - x^\sigma)^{\alpha} t^{\sigma(1-\alpha-\eta)-1} f(t)\, dt, \ -1 < \alpha < 0,$$

where $a < x < b, \quad \sigma > 0.$

From the theory of Abel integral equations it follows that the inverse operators are given by

(8) $$I^{-1}_{\eta,\alpha}(a, b;\sigma) f(x) = I_{\eta+\alpha,-\alpha}(a, b;\sigma) f(x)$$

(9) $$K^{-1}_{\eta,\alpha}(c, d;\sigma) f(x) = K^{-1}_{\eta+\alpha,-\alpha}(c, d;\sigma) f(x).$$

We require two lemmas given by Lowndes [4] which define the pairs of operators:

Lemma A. Let $I_{\eta,\alpha}(a, b;\sigma)$, $I^{-1}_{\eta,\alpha}(d, x;\sigma)$ be operators as defined earlier, then

(10) $$I^{-1}_{\eta,\alpha}(d, x;\sigma) I_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma \sin(\alpha\pi)}{\pi} \frac{x^{-\sigma\eta}}{(x^\sigma - d^\sigma)} \int_a^b \frac{t^{\sigma(\eta+1)-1}(d^\sigma - t^\sigma)^\alpha}{(x^\sigma - t^\sigma)} f(t)\, dt, \text{ provided } x > d \geq b > a.$$

Lemma B. Let $K_{\eta,\alpha}(a, b;\sigma)$, $K^{-1}_{\eta,\alpha}(x,d;\sigma)$ be operators as defined earlier, then

(11) $$K^{-1}_{\eta,\alpha}(c, d;\sigma) K_{\eta,\alpha}(a, b;\sigma) f(x) = \frac{\sigma \sin(\alpha\pi)\, x^{\sigma(\alpha+\eta)}}{\pi \quad (d^\sigma - x^\sigma)} x \int_a^b \frac{t^{\sigma(1-\alpha-\eta)-1}(t^\sigma - d^\sigma)^\alpha}{(t^\sigma - x^\sigma)} f(t)\, dt, \text{ provided } x < d \leq a < b.$$

Two well-known results [4] which play an important part in our solution are :

(12) $$M[I_{\eta,\alpha}(0, x;\sigma) f(x) ; s] = \frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)} M[f(x) ; s],$$

(13) $$M[K_{\eta,\alpha}(x,\infty;\sigma) f(x) ; s] = \frac{\Gamma(\eta+s/\sigma)}{\Gamma(\eta+\alpha+s/\sigma)} M[f(x) ; s].$$

In what follows we are concerned with n-ranges of the variable x,

$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\}$ Hence, we have

$$y_0 \in \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap F(T) \cap F(U) \subset \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

So from Lemma 4, we have

$$\{y_0\} = \bigcap_{n=1}^{\infty} \overline{co}\,\{U_m x : m \geq n\} \cap \bigcap_{n=1}^{\infty} F(T_n).$$

Hence, we obtain $x_n \rightarrow y_0 \in F(T) \cap F(U)$.

REFERENCES

[1] F.E. Browder, Nonlinear operators and nonlinear equation of evolution in Banach spaces, *Amer. Math. Soc.* **18** (1976).

[2] R.E. Bruck, On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces, *Israel J. Math.* **38** (1981), 304-314

[3] D. Van Dulst, Equivalent norms and the fixed point property for nonexpansive mappings, *J. London. Math. Soc.* **25** (1982), 139-144.

[4] J.P. Gossez and E. Lami Dozo, Some geometric properties related to the fixed point theory for nonexpansive mappings, *Pacific. J. Math.* **40** (1972), 565-573.

[5] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.* **4** (1953) 506-510.

[6] Z. Opial, Weak convergence of the sequence of successive approximate for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591-597.

[7] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, *Math. Anal. Appl.* **67** (1979), 274-276.

[8] W. Takahashi and G.E. Kim, Approximating fixed points of nonexpensive mappings Banach spaces, *Mathematica Joponica* (to appear).

[9] H.K. Xu, 'Inequalities in Banach spaces with applications, *Nonlinear Anal.* **16** (1991) 1127-1138.

Jñānābha, Vol. 30, 2000

SOME n-TUPLE INTEGRAL EQUATIONS INVOLVING INVERSE MELLIN TRANSFORMS

By

A.P. Dwivedi, Jyotsna Chandel and Poonam Bajpai

Department of Mathematics

H.B. Technological Institute, Kanpur-208002, Uttar Pradesh, India

(Received : May 23, 2000)

ABSTRACT

The solution of n–tuple integral equations when n is even involving inverse Mellin transform has been obtained by reducing them to a system of Fredholm integral equations.

1. Introduction. In this paper we solve the n-tuple integral equations when n is even

(1) $$M^{-1}\left[\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\,\phi(s)\,;x\right] = f_i(x),\ a_{i-1} < x < a_i,\ (i = 1,3,\dots, n-1)$$
$$\text{and } a_0 = 0,$$

(2) $$M^{-1}\left[\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\,\phi(s)\,;x\right] = g_i(x),\ a_{i-1} < x < a_i,\ (i = 2,4,\dots, n)$$
$$\text{and } a_n = \infty,$$

where $\alpha, \beta, \xi, \eta, \delta, > 0$, $\sigma > 0$ are real parameters, g_i $(i = 2, 4 \dots, n)$ are known functions, $\phi(s)$ is to be determined and

(3) $$M[h(x)\,;s] = H(s),$$
$$M^{-1}[H(s)\,;x] = h(x),$$

denote the Mellin transform of $h(x)$ and its inversion formula respectively.

The above equations are an extension of the triple integral equations solved by Lowndes [4], quardruple integral equations solved by Dwivedi, Kushwaha and Trivedi [8], and six integral equations solved by Dwivedi, Shukla and Shukla [9] by means of a systematic applications of some slightly extended forms of Erdèlyi-Köber operators of fractional integration [2].

Using the properties of some slightly extended forms of the Erdèlyi-Köber operators given in [4]. We show in purely formal manner that the solution of the n-tuple integral equations can be expressed in terms of solution of Fredholm intergral equation of the second kind. The mothod of solution employed here will be seen to follow closely that used by Ahmad [7] to obtain the solution of som quadruple integral equation involving Bessel functions.

namely

(14) $I_i = \{x : a_{i-1} \le x < a_i\}$, $(i = 1,2,3,..,n)$ *and* $(a_0 = 0,\ a_n \to \infty)$

and we shall write any function $f(x)$, $x \ge 0$, in the form

(15) $$f(x) = \sum_{i=1}^{n} f_i(x),$$

where

(16) $$f_i(x) = \begin{cases} f(x), & x \in I_i \\ 0, & \text{otherwise} \end{cases} \quad i = 1,2,3,\ldots, n.$$

and similarly for g.

3. Solution of the Integral Equations. Using the notation of equations (15) and (16) we can write the n-tuple integral equations (when n is even) (1) and (2) as

(17) $$M^{-1}\left[\frac{\Gamma(1+\eta-s/\sigma)}{\Gamma(1+\eta+\alpha-s/\sigma)}\,\phi(s)\,;\,x\right] = f(x),$$

(18) $$M^{-1}\left[\frac{\Gamma(\xi+s/\delta)}{\Gamma(\xi+\beta+s/\delta)}\,\phi(s)\,;\,x\right] = g(x),$$

where $f_1, f_3, \ldots, f_{n-1}$ and $g_2, g_4, \ldots, g_n$ are prescribed functions while f_2, $f_4, \ldots, f_n$ and $g_1, g_3, \ldots, g_{n-1}$ are unknown functions to be determined. If we write

(19) $\phi(s) = M[\phi(x); s]$,

and use the formulae (12) and (13) we find that equations (17) and (18) assume the operational form

(20) $I_{\eta,\alpha}(0, x; \sigma)\,\phi(x) = f(x)$

(21) $K_{\xi,\beta}(x,\infty\,; \delta)\,\phi(x) = g(x)$.

Using the formulae (8) and (9) and solving the above equations for $\phi(x)$ be obtain

(22) $\phi(x) = I_{\eta,\alpha}(0, x; \sigma)\; f(x)$

(23) $\phi(x) = K_{\xi+\beta,-\beta}(x,\infty\,; \delta)\; g(x)$.

We proceed to determine ϕ. The subscripts on all the operators I's will be supposed to have the subscripts $(\eta,\alpha;\sigma)$ understand and all K's to have subsript $(\xi, \beta; \delta)$. Evaluating (22) on $I_1, I_2, I_3, \ldots, I_{n-1}$ we get

(24) $$\phi_1 = \binom{x}{a_0} I^{-1} f_1,$$

(25) $$\phi_2 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{x}{a_1} I^{-1} f_2,$$

(26) $$\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \binom{x}{a_2} I^{-1} f_3,$$

... ...

... ...

(27) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} f_2 + \ldots + \binom{x}{a_{n-2}} I^{-1} f_{n-1},$

where $a_0 = 0$

Evaluating (23) on $I_2, I_3, \ldots, I_n$ we get

(28) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} g_3 + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(29) $\phi_3 = \binom{a_3}{x} K^{-1} g_3 + \binom{a_4}{a_3} K^{-1} g_4 + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

... ...

... ...

(30) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(31) $\phi_{n-1} = \binom{a_{n-1}}{x} K^{-1} g_{n-1} + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

(32) $\phi_n = \binom{a_n}{x} K^{-1} g_n.$

where $a_n \to \infty$.

Since f_1 and g_n are known, ϕ_1 and ϕ_n can be determined by equations (24) and (32).

We now solve (25) for f_2 and substitute its value in (26) to get

(33) $\phi_3 = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I\,[\,\phi_2 - \binom{a_1}{a_0} I^{-1} f_1\,] + \binom{a^3}{a_2} I^{-1} f_3.$

Similarly

(34) $\phi_{n-1} = \binom{a_1}{a_0} I^{-1} f_1 + \binom{a_2}{a_1} I^{-1} \binom{x}{a_1} I\,[\,\phi_2 - \binom{a_1}{a_0} I^{-1} f_1\,] + \binom{a_3}{a_2} I^{-1} f_3 + \ldots$

$+ \binom{a_{n-1}}{a_{n-2}} I^{-1} f_{n-1},$

We solve equation (31) for g_{n-1} and substitute its value in (30) to get

(35) $\phi_{n-2} = \binom{a_{n-2}}{x} K^{-1} g_{n-2} + \binom{a_{n-1}}{a_{n-2}} K^{-1} \binom{a_{n-1}}{x} K\,[\,\phi_{n-1} - \binom{a_n}{a_{n-1}} K^{-1} g_n\,]$

$+ \binom{a_n}{a_{n-1}} K^{-1} g_n.$

Similarly from equation (28) we find out the value of ϕ_2

(36) $\phi_2 = \binom{a_2}{x} K^{-1} g_2 + \binom{a_3}{a_2} K^{-1} \binom{a_3}{x} K\,[\,\phi_3 - \binom{a_4}{a_3} K^{-1} g_4 - \ldots - \binom{a_n}{a_{n-1}}$

$K^{-1} g_n] + \ldots + \binom{a_n}{a_{n-1}} K^{-1} g_n.$

We have thus arrived at n–tuple simultaneous equations (24), (36), (33), ... , (35), (34) and (32) associated with n-tuple unknown functions $\phi_1, \phi_2, \phi_3, \ldots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n respectively.

From these equations we can obtain the values of these unknowns and hence the solution of the problem will be determined by equation (19).

4. An Application

We shall consider now the n-tuple integral equations, when n is even which are extension of triple integral equations solved by Lowndes [4] :

(37) $\int_0^\infty u^{-2n}\Psi(u)J_{2q}(ux)du = F_i(x),\ a_{i-1} < x < a_i,\ (i = 1,3,\dots n-1)$ *and* $a_0=0$

(37) $\int_0^\infty \Psi(u)J_{2q}(ux)du = 0,\ a_{i-1} < x < a_i,\ (i = 2,4,\dots n)$ *and* $a_n \to \infty$

where $J_{2p}(ux)$ is the Bessel function of the first kind of order $2p$, $F_1(x)$, $F_3(x),\dots, F_{n-1}(x)$ are prescribed functions and $\Psi(u)$ is to be determined. When $p = q$ and $a_{i>3,\dots n} \to \infty$ these are equations investigated by Ahmad [7]. We now show, in a fairly straight forward manner, that the above equations can be transformed into equations of the type (1) to (2) with $g_i = 0,\ (i = 2,4,\dots n,)$

Denoting the Mellin transforms of $\Psi(u)$ by

(39) $M[\Psi(u); s] = \Psi(s)$

and using the result

(40) $$M[\xi^{-2n} j_{2q}(\xi); s] = 2^{s-1-2n}\frac{\Gamma(q-n+s/2)}{\Gamma(1+n+q-s/2)},$$

we have, on applying the Faltung theorem for Mellin transforms that the integral equations (37) and (38) can be written in the form

(41) $$M^{-1}\left[\frac{\Gamma(1+p-s/2)}{\Gamma(1+n+q-s/2)}\phi(s); x\right] = 2^{1+2n}x^{-2n}F_i(x),\ a_{i-1} < x < a_i,\ (i = 1,3\dots n-1) \text{ and } a_0 = 0,$$

(41) $$M^{-1}\left[\frac{\Gamma(p+s/2)}{\Gamma(q-n+s/2)}\phi(s); x\right] = 0,\ a_{i-1} < x < a_i,\ (i = 2,4,\dots,n) \,\&\, a_0 \to \infty$$

where

(43) $$\phi(s) = 2^s\left[\frac{\Gamma(q-n+s/2)}{\Gamma(1+p-s/2)}\right]\Psi(1-s).$$

These are the same as equations (1) and (2) with

$\sigma = \delta = 2,\ \xi = \eta = p,\ \alpha = q-p+n,\ \beta = q-p-n,$

and

(44) $$f_i(x) = 2^{1+2n}x^{-2n}F_i(x),\ i = 1,3,\dots, n-1$$
$$g_i = 0,\ i = 2,4,\dots, n.$$

Using the results of the previous section we have therefore, that the solution of equations (41) and (42) can be found in terms of a function $\phi(x)$ by

(45) $\phi(s) = M[\phi(x); s]$,

where the functions $\phi_1, \phi_2, \phi_3, \dots, \phi_{n-2}, \phi_{n-1}$ and ϕ_n are obtained from

equations (24), (36), (33), ..., (35), (34) and (32) with parameters ξ, η etc. given by equations (44).

Finally, in order to find the solution of the integral equations (37) and (38) in terms of $\phi(x)$, we proceed in the following way.

From equation (39) we have that the solution is

$$(46)\quad \psi(u) = M^{-1}[\psi(s); u],$$

$$= M^{-1}\left[2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)} M\{\phi(x); 1-s\}; u\right],$$

on using equations (43) and (44). Inverting the order of integration in the last equation we get

$$(47)\quad \psi(u) = \int_0^\infty \phi(x) M^{-1}\left[2^{s-1}\frac{\Gamma(1/2+p+s/2)}{\Gamma(1/2+q-n-s/2)}; ux\right] dx,$$

$$= \int_0^\infty \left(\frac{ux}{2}\right)^{1+n+p-q} \phi(x) J_{p+q-n}(ux)\, dx.$$

after applying the results (40). When $p = q$, this solution is exactly the same as that found by Ahmad [7].

REFERENCES

[1] A. Erdélyi, Some dual integral equations, *SIAM, J. Appl. Math.*, **16** (1968), 1338.

[2] I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North Holland, (1966).

[3] I.N. Sneddon, *Fourier Transforms*, New York, (1951).

[4] J.S. Lowndes, Some triple integral equations, *Pac. J. Math.*, **38** (1971), 515.

[5] J.C. Cooke, The solutions of triple and quadruple integral equations and Fourier Bessel series, *Quart. J. Mech. Appl. Math.*, **25** (1972), 247.

[6] J.C. Cooke, The solution of triple integral equations in operational form *Quart. J. Mech. Appl. Math.*, **18** (1965), 57.

[7] M.I. Ahmad, Quadruple integral equations and operators of fractional integration', *Glasgow Math. J.*, **12** (1971), 60.

[8] A.P. Dwivedi, S.P. Kushwaha and T.N. Trivedi, Some quadruple integral equations, *JMACT*, **13** (1980), 35.

[9] A.P. Dwivedi, B.D. Shukla and S.C. Shukla, Some six integral equations involving inverse Mellin transforms, *JMACT*, **19** (1986), 107.

Jñānābha, Vol. 30, 2000

A UNIFIED PRESENTATION OF CERTAIN SUBCLASSES OF STARLIKE AND UNIFORMLY CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

BY

KAUSHAL KISHORE DIXIT AND INDU BALA Misra

Department of Mathematics,

Janta College, Bakewar - 206124 (Etawah), Uttar Pradesh, India

(Received : October 15, 2000)

ABSTRACT

A unified class $T(\alpha,\beta)$ of starlike and uniformly convex functions in the open disk has been introduced. Number of theorems involving for example, sharp distortion inequalities and modified Hamdard Product (or convolution) of functions belonging to the class $T(\alpha,\beta)$ have been obtained. It is also shown how these theorems would apply to yield various results given in the literature.

1. Introduction. Let T denote the class of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ a_k \geq 0 \tag{1}$$

Definition 1. The Class UCT. A function $f(z)\ \varepsilon\ T$ given by (1) is said to be uniformly convex, if it satisfies the inequality

$$1 + Re\,\{z f''(z)/f'(z)\} \geq |zf''(z)/f'(z)|,\ z\ \varepsilon\ U \tag{2}$$

We denote the class of such functions by UCT.

Definition 2. The Class $UCT\ (\alpha)$. A function $f(z)$ of the form (1) is said to be in $UCT\ (\alpha)$, $\alpha \geq 0$ if and only if

$$1+Re\,\{z f''(z)/f'(z)\} \geq \alpha\,|z f''(z)/f'(z)|,\ z\ \varepsilon\ U \tag{3}$$

Remark:

$UCT\,(1) = UCT$

$UCT\,(0) = C$, the class of convex functions with negaive coefficients

Inclusion Results.

$UCT\,(\alpha)$	$\subseteq$	UCT	For $\alpha \geq 1$
UCT	$\subseteq$	$UCT\,(\alpha)$	For $0 < \alpha \leq 1$
$UCT\,(\alpha)$	$\subseteq$	$UCT\,(\gamma)$	For $\alpha \geq \gamma$

These results are straight forward from the definition.

Definition 3. The Class TS_p. A function $f(z)$ of the form (1) is said to be in TS_p if and only if

$$|zf'(z)/f(z) - 1| \leq Re\{zf'(z)/f(z)\} \; z \,\varepsilon\, U \tag{4}$$

Definition 4.The Class $TS_p(\alpha)$. A function $f(z)$ of the form (1) is said to be in $TS_p(\alpha)$ if and only if

$$\alpha\,|zf'(z)/f(z) - 1| \leq Re\{zf'(z)/f(z)\} \; z \,\varepsilon\, U \tag{5}$$

Remark.

$TS_p(1) = TSp$

$TS_p(0) = T$

Inclusion Results.

$TS_p(\alpha) \subseteq TS_p$	For	$\alpha \geq 1$
$TS_p \subseteq TS_p(\alpha)$	For	$0 \leq \alpha \leq 1$
$TS_p(\alpha) \subseteq TS_p(\gamma)$	For	$\alpha \geq \gamma$

These results follow easily from definition.

The classes $UCT(\alpha)$ and $TS_p(\alpha)$ were introduced by Murugusundaramurthy [1].

Murugusundaramurthy [1] gave the following lemmas :

Lemma 1. A function $f(z)$ defined by (1) is in $UCT(\alpha)$, if and only if

$$\sum_{k=2}^{\infty} k\,[k(\alpha+1) - \alpha]\, a_k \leq 1 \tag{6}$$

The result (6) is *SHARP* for functions

$f_k(z) = z - z^k / k(\alpha+1) - \alpha$.

Lemma 2. A function $f(z)$ defined by (1) is in $TS_p(\alpha)$, if and only if,

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\, a_k \leq 1. \tag{7}$$

The result (7) is *SHARP* for functions

$f_k(z) = z - z^k / [k(\alpha+1) - \alpha]$.

In view of Lemma 1 and Lemma 2, it would seem to be natural to introduce and study an interesting unification of the class $UCT(\alpha)$ and $TS_p(\alpha)$.

Thus we say that a function $f(z)$ defined by (1) belongs to the class $T(\alpha, \beta)$ if and only if

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,[1 - \beta + \beta k]\, a_k \leq 1 \tag{8}$$

for some $\alpha \geq 0$ and some $\beta\ (0 \leq \beta \leq 1)$.

Since,

$1 - \beta + \beta k \geq 1 \qquad (0 \leq \beta \leq 1)$

$\qquad\qquad\qquad (k = 2, 3, ...)$

Therefore,

$T(\alpha, 0) = TS_p(\alpha)$,

and $T(\alpha, 1) = UCT(\alpha)$.

The object of this paper is to present a unified study of the $UCT(\alpha)$ and $TS_p(\alpha)$ by proving various interesting properties and characteristics of the general class $T(\alpha, \beta)$.

2. Distortion Inequalities.

Theorem 1. If a function $f(z)$ defined by (1) is in the class $T(\alpha, \beta)$, then

$$r - \frac{r^2}{(\alpha+2)(1+\beta)} \leq |f(z)| \leq r + \frac{r^2}{(\alpha+2)(1+\beta)}. \tag{9}$$

and

$$1 - \frac{2r}{(\alpha+2)(1+\beta)} \leq |f(z)| \leq 1 + \frac{2r}{(\alpha+2)(1+\beta)}, \quad |z| = r. \tag{10}$$

The estimates are sharp.

Proof. Since,

$$(a+2)(1+\beta) \sum_{k=2}^{\infty} a_k \leq \sum_{k=2}^{\infty} [k(\alpha+1)-\alpha][1-\beta+\beta k]\, a_k \leq 1,$$

we have,

$$\begin{aligned} |f(z)| &\leq |z| + |z|^2 \sum_{k=2}^{\infty} a_k \\ &\leq |z| + \frac{1}{(\alpha+2)(1+\beta)} |z|^2 \\ &\leq r + \frac{r^2}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f(z)| \geq r - \dfrac{r^2}{(\alpha+2)(1+\beta)}$.

This proves the estimate (9) of the theorem 2.

We note that,

$$\begin{aligned} \frac{(\alpha+2)(1+\beta)}{2} \sum_{k=2}^{\infty} a_k &\leq \sum_{k=2}^{\infty} [1+\alpha/k][1-\beta+\beta k]k\, a_k \\ &\leq \sum_{k=2}^{\infty} \frac{[k(\alpha+1)-\alpha]}{k} [1-\beta+\beta k]k\, a_k \\ &\leq 1. \end{aligned}$$

Now we have

$$\begin{aligned} |f'(z)| &\leq 1 + |z| \sum_{k=2}^{\infty} k\, a_k \\ &\leq 1 + \frac{2}{(\alpha+2)(1+\beta)} |z| \\ &\leq 1 + \frac{2r}{(\alpha+2)(1+\beta)}. \end{aligned}$$

Likewise $|f'(z)| \geq 1 - \dfrac{2r}{(\alpha+2)(1+\beta)}$.

$$\Rightarrow \qquad \gamma\,(k+1) - k \;\le\; [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]$$

$$\Rightarrow \qquad \gamma \;\le\; \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - k}{\gamma+1}\;.$$

If we set

$$\phi\,(k) = \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - k}{\gamma+1}\;.$$

We easily see that $\phi\,(k)$ is an increasing functionj of k.
on setting $k = 2$, we get

$$\gamma \;\le \phi\,(2) = (\alpha+2)^2\,(1-\beta) - 2.$$

This completes the proof.

Finally taking the function given by

$$f_j(z) = z - \frac{1}{(\alpha+2)\,(1+\beta)}\,z^2.$$

The result of the theorem 2 is *SHARP*.

Remark. Putting $\beta = 0$ and $\beta = 1$ in the theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT\,(\alpha)$ given earlier by Murugusundaramurthy [1].

Theorem 3. Let the function $f_j(z)$ $(j = 1,2)$ be in the class $T(\alpha,\beta)$, then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} (a_{1,k}^2 + a_{2,k}^2)\,z^k$$

belongs to the class $T\,(\gamma,\beta)$, where

$$\gamma = \frac{(\alpha+2)^2\,(1+\beta) - 4}{2}\,.$$

Proof. Since,

$$\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,a_{j,k}^2$$

$$\le \Big[\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]\,[1-\beta+\beta k]\,a_{j,k}\,\Big]^2 \le 1\;(j = 1,2)$$

we have,

$$\sum_{k=2}^{\infty} [\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,[a_{1,k}^2 + a_{2,k}^2\,] \le 2.$$

Thus, it is sufficient to find a largest γ such that

$$[k\,(\gamma+1) - \gamma]\,[1-\beta+\beta k] \le [k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k]^2\,/\,2\,,$$

that is,

$$\gamma\,(k-1) - k \le [k\,(\alpha+1) - \alpha]^2\,[1-\beta+\beta k]\,/\,2,$$

$$\gamma\,(k-1) \;\le\; \frac{[\,k\,(\alpha+1) - \alpha\,]^2\,[1-\beta+\beta k] - 2k}{2}$$

This proves the estimate (10) of the theorem 2.

The bounds are *SHARP* since the equalities are attained for the function

$$f(z) = z - \frac{1}{(\alpha+2)(1+\beta)} z^2 .$$

Remark. Putting $\beta = 0$ and $\beta = 1$ in theorem 2, we obtain the corresponding results for the classes $TS_p(\alpha)$ and $UCT(\alpha)$ given earlier by Murugusundaramurthy [1].

3. Modified Hadamard Product (Convolution Product)

Definition. Let the function $f_j(z)$ $(j = 1,2,)$ ε T is of the form

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{j,k} z^k, \; (a_{j,k} \geq 0, j = 1,2) \tag{11}$$

The modified Hadamard Product (or convolution) of $f_1(z)$ and $f_2(z)$ is denoted by $f_1 * f_2(z)$ and defined by

$$f_1 * f_2(z) = z - \sum_{k=2}^{\infty} a_{1,k}\, a_{2,k}\, z^k \tag{12}$$

Theorem 2. Let the $f_j(z)$ $(j = 1,2)$ be in the class $T(\alpha,\beta)$. Then $f_1 * f_2(z)$ belongs to the class $T(\gamma,\beta)$, where $\gamma = (\alpha+2)^2(1+\beta) - 2$.

Proof. Applying the technique used earlier by Schild and Silverman [2], we need to find the largest γ such that

$$\sum_{k=2}^{\infty} [k(\gamma+1) - \gamma]\,[1-\beta+\beta k]\, a_{1,k}\, a_{2,k} \leq 1.$$

Since $f_j(z)$ $(j= 1,2)$ ε $T(\alpha,\beta)$, therefore, we have

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,[1-\beta+\beta k]\, a_{j,k} \leq 1 \; (j = 1,2).$$

Therefore, by Cauchy Schwarz inequality, we have

$$\sum_{k=2}^{\infty} [k(\alpha+1) - \alpha]\,[1-\beta+\beta k]\sqrt{a_{1,k}\, a_{2,k}} \leq 1 \tag{13}$$

Thus, it is sufficient to show that

$$[k(\gamma+1) - \gamma]\,[1-\beta+\beta k]\, a_{1,k}\, a_{2,k} \leq [k(\alpha+1) - \alpha]\,[1-\beta+\beta k]\sqrt{(a_{1,k}\, a_{2,k})}$$

that is,

$$\sqrt{(a_{1,k}\, a_{2,k})} \leq \frac{k(\alpha+1)-\alpha}{k(\gamma+1)-\gamma}$$

that is, if

$$\frac{1}{[k(\alpha+1)-\alpha]\,[1-\beta+\beta k]} \leq \frac{k(\alpha+1)-\alpha}{k(\gamma+1)-\gamma} \quad \text{(from (13))}$$

that is, if $\quad [k(\gamma+1) - \gamma] \leq [k(\alpha+1) - \alpha]^2\,[1-\beta+\beta k]$

that is,

$$\gamma \leq \frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-2k}{2\,(k-1)} .$$

If we set,

$$\Psi\,(k) = \frac{[\,k\,(\alpha+1)-\alpha\,]^2\,[1-\beta+\beta k]-2k}{2\,(k-1)} .$$

We easily see that $\Psi\,(k)$ is an increasing function of k. On setting $k = 2$, we get,

$$\gamma \leq \Psi\,(z) = \frac{(\alpha+2)^2\,(1+\beta)-4}{2} .$$

This completes the proof.

Remark. Taking $\beta = 0$ and $\beta = 1$ we get the corresponding results for the classes $TS_p(\alpha)$ and $UCT\,(\alpha)$ given earlier by Murugusundaramurthy [1].

REFERENCES

[1] G. Murugusundaramurthy, *Studies on Classes of Analytic Functions with Negative Coefficients*, Ph. D. Thesis, University of Madras (1994).

[2] A Schild and H. Silverman, Convolution of Univalent Functions with Negative Coefficients, *Annals, Univ. Marie-Curie*, **29** (1975), 99-107.

Jñānābha, Vol. 30, 2000

SOME RESULTS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION, GENERALIZED PROLATE SPHEROIDAL WAVE FUNCTION, GENERALIZED POLYNOMIALS AND THE MULTI-VARIABLE *H*-FUNCTION

By

V.G. Gupta

Department of Mathematics

University of Rajasthan, Jaipur-302 004, Rajasthan

and

Suman Jain

S.S. Jain Subodh Post Graduate College, Jaipur-302 004, Rajasthan

(Received : August 30, 2000)

ABSTRACT

In this paper an integral involving generalized prolate spheroidal wave function, generalized hypergeometric function, the generalized polynomials and the (Srivastava and Panda) H-function of several complex variables has been evaluated and an expansion formula for the product of the generalized hypergeometric function, generalized polynominals and the H-function of several complex variables has been established with the application of this integral. On account of the most general nature of the functions and the polynomials occuring in these results, our finding provide interesting unifications and extensions of a large number of (new an known) results. We record here only two special cases of our main integral, first involves the product of the generalized prolate spheroidal wave fuction, the generalized hypergeometric function, the generalized polynominals and the generalized Lauricella function of several complex variables ([13], p. 454), the other one involves the product of the generalized spheroidal wave function, the generalized hypergeometric function, the general class of polynomials and the multivariable H-function. With the application of the above two special cases of integrals we can also derive two expansion formulae for the product of the functions and polynomials stated in the integrals respectively. Out of several known results which follow as special cases of our main integral and expansion formula we refer here only to the results of Mishra [7], Gupta [6],

$$, \dots y_R(1-x)^{g_R}(1+x)^{w_R}]\ H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r(1-x)^{h_r}(1+x)^{k_r}]$$

$$= \sum_{t,p,q=0}^{\infty} 2^{\rho+\sigma-\alpha-\beta+gq+wq}\, L(y_1,\dots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q}$$

$$\cdot \sum_{m}^{t+p} \frac{(-t-p)_m(\alpha+\beta+t+n+p+1)_m}{m!\,(\alpha+1)_m}\, H^{0,\lambda+2:\,(u',v');\dots;(u^{(r)},v^{(r)})}_{A+2,C+1:\,(B',D');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-gq-g_1\alpha_1-\dots-g_R\alpha_R: h_1,\dots,h_r],\ [-\sigma-wq-w_1\alpha_1-\dots-w_R\alpha_R: k_1,\dots,k_r],\\ [(c):\psi',\dots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\dots-(g_R+w_R)\alpha_R: h_1+k_1,\dots,h_r+k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1 2^{h_1+k_1},\dots, z_r 2^{h_r+k_r}\right)$$

$$\left\{\phi_t^{\alpha,\beta}(s,x)/(R_{p,t}^{\alpha,\beta}(s))^2\, \frac{\Gamma(t+\alpha+p+1)\Gamma(t+p+\beta+1)}{(2t+2p+\alpha+\beta+1)\Gamma(t+p+1)\Gamma(t+p+\alpha+\beta+1)}\right\} \quad \dots (3.1)$$

where $L(y_1, \dots y_R)$ is same as given in (2.2), $Re\ \rho + \left(\sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$, $Re\left(\sigma + \sum_{i=1}^{r} k_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1$, $h_i, k_i > 0\ \forall i \in \{1,\dots,r\}$; $g > 0$, $w = 0$, $g_{j'} > 0, w_{j'} > 0$, $\forall j' \in \{1,\dots,R\}$; $p_{j'}$ $(j' = 1,\dots,R)$ is an arbitrary positive integer and the coefficients $A\,[q_1,\alpha_1;\dots;q_R,\alpha_R]$ are arbitrary constants, real or complex, $T_i > 0$, $|arg\, z_i| < T_i\pi/2$, $i = 1,\dots,r$, $j = 1,\dots,u^{(i)}$, $\alpha > -1$, $\beta > -1$, $M \le N$ ($M = N+1$ and $|y| < 1$).

Proof. Let

$$f(x) = (1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\ {}_MF_N\left[\begin{matrix}e_M\\ f_N\end{matrix}; y(1-x)^g(1+x)^w\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$$, \dots y_R(1-x)^{g_R}(1+x)^{w_R}]\ H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r(1-x)^{h_r}(1+x)^{k_r}]$$

$$= \sum_{t=0}^{\infty} E_t\, \phi_t^{\alpha,\beta}(s,x), \qquad (-1 < x < 1) \quad \dots (3.2)$$

Equation (3.2) is valid since $f(x)$ is continuous and of bounded variation in the open interval $(-1,1)$.

Now mutliply both side of (3.2) by

$(1-x)^{\alpha}(1+x)^{\beta}\, \phi_n^{\alpha,\beta}(s,x)$

and integrate with respect to x from -1 to 1. Change the order of integration and summation (which is permissible) on the right, we obtain

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, \phi_n^{\alpha,\beta}(s,x)\ {}_MF_N\left[\begin{matrix}e_M\\ f_N\end{matrix}; y(1-x)^g(1+x)^w\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}$$

Srivastava and Singh [16].

1. Introduction. The solution of the following differential equation

$$(1-x^2)\frac{d^2y}{dx^2}+[(\beta-\alpha)-(\alpha+\beta+2)\,x]\frac{dy}{dx}+[\xi(s)-s^2x^2]\,y=0 \quad ...(1.1)$$

is the generalized prolate spheroidal wave function (Gupta [5]) which has been denoted as

$$\phi_n^{\alpha,\beta}(s,x)=\sum_{p=0}^{\infty} R_{p,n}^{\alpha,\beta}(s)\,P_{p+n}^{(\alpha,\beta)}(x) \quad ...(1.2)$$

where $s=0$ and $\xi(0)=(n+p)(\alpha+\beta+n+p+1),\ p\geq 0$.

The generalized polynomials (multivariable) defined by srivastava [11] represented in the following manner :

$$S_{q_1,\dots,q_R}^{p_1,\dots,p_R}\begin{bmatrix}x_1\\ \vdots\\ x_R\end{bmatrix}=S_{q_1,\dots,q_R}^{p_1,\dots,p_R}[x_1,\dots,x_R]$$

$$=\sum_{\alpha_1=0}^{[q_1/p_1]}\cdots\sum_{\alpha_R=0}^{[q_R/p_R]}\frac{(-q_1)p_1\alpha_1}{\alpha_1!}\cdots\frac{(-q_R)p_R\alpha_R}{\alpha_R!}.\,A\,[q_1,\alpha_1;\dots;q_R,\alpha_R]$$

$$x_1^{\alpha_1}\dots x_R^{\alpha_R} \quad ...(1.3)$$

where $q_j=0,\ 1,\ 2,\ \dots;\ p_j\ (j=1,\ \dots,\ R)$ are non-zero arbitrary positive integer. The coefficients. $A\,[q_1,\alpha_1;\dots;q_R,\alpha_R]$ being arbitrary constants, real or complex. If we take $R=1$ in the equation (3) and denote $A\,[q,\alpha]$ thus obtained by $A_{q,\alpha}$, we arrive at the well known general class of polynomials $S_q^p\,[x]$ intriduced by Srivastava [12].

The multivariable H-function of several complex variables occuring in the paper will be defined by Srivastava and Panda ([14], [15]) by means of the multiple Mellin-Barnes integral.

$$H\,[z_1,\dots,z_r]=$$

$$=H_{A,\ C:\ (B',D');\dots;(B^{(r)},D^{(r)})}^{0,\lambda\,:\,(u',\,v');\,\dots\,;\,(u^{(r)},\,v^{(r)})}\left(\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}];\\ [(c):\psi',\dots,\psi^{(r)}]:[(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1,\dots,z_r\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\dots\phi_r(\xi_r)\,\psi(\xi_1,\dots,\xi_r)\,z_1^{\xi_1}\dots z_r^{\xi_r}\,d\xi_1\dots d\xi_r,\ \omega=\sqrt{-1} \quad ...(1.4)$$

For the convergence of the integral given by (1.4) and other details of the multivariable H-function, we refer to the book by Srivastava et. al. ([10], p.251-3, eqn. (C-1)-(C-5)).

The orthogonality property of the generalized prolate spheroidal wave function (Gupta [5], p. 107, eqn. (3.1).

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}\,\phi_n^{\alpha,\beta}(s,x)\,\phi_t^{\alpha,\beta}(s,x)\,dx=N_{n,t}^{\alpha,\beta}\,\delta_{n,t} \quad ...(1.5)$$

where

$$N_{n,t}^{\alpha,\beta}=\sum_{p=0}^{\infty}(R_{p,n}^{\alpha,\beta}(s))^2\,2^{\alpha+\beta+1}\frac{\Gamma(n+\alpha+p+1)\Gamma(n+p+\beta+1)}{(2n+2p+\alpha+\beta+1)\Gamma(n+p+1)\Gamma(n+p+\alpha+\beta+1)} \quad ...(1.6)$$

and $\delta_{n,t}$ is the kronecker delta.

2. The main integral.

The following integral has been evaluated in this paper :

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, \phi_n^{\alpha,\beta}(s,x)\ {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix} ; y(1-x)^{g}(1+x)^{w}\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}$$

$$(1+x)^{w_1}, \dots y_R(1-x)^{g_R}(1+x)^{w_R}]\, H\,[z_1(1-x)^{h_1}(1+x)^{k_1}, \dots z_r(1-x)^{h_r}(1+x)^{k_r}]dx$$

$$= \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq} L(y_1,\dots,y_R)\, R^{\alpha,\beta}_{p,n}(s)\ \frac{(e_M)_q\, y^q}{q!\,(f_N)_q}$$

$$\sum_{m}^{n+p} \frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m}\, H^{0,\lambda+2:(u',v');\dots;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');\dots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho-gq-g_1\alpha_1-\dots-g_R\alpha_R: h_1,\dots,h_r],\ [-\sigma-wq-w_1\alpha_1-\dots-w_R\alpha_R: k_1,\dots,k_r], \\ [(c):\psi',\dots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\dots-(g_R+w_R)\alpha_R: h_1+k_1,\dots,h_r+k_r]: \end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\delta'];\dots;[d^{(r)}:\delta^{(r)}]; \end{matrix}\ z_1 2^{h_1+k_1},\dots, z_r 2^{h_r+k_r}\right) \quad \dots (2.1)$$

where

$$L(y_1,\dots,y_R) = \sum_{\alpha_1=0}^{[q_1/p_1]} \dots \sum_{\alpha_R=0}^{[q_R/p_R]} \prod_{j'=1}^{R} \left[\frac{(-q_{j'})_{p_{j'}\alpha_{j'}}}{\alpha_{j'}!}\, y_j^{\alpha_j}\, 2^{(g_{j'}+w_{j'})\alpha_{j'}}\right.$$

$$A\,[q_1,\alpha_1;\dots;q_R,\alpha_R] \quad \dots (2.2)$$

provided that $h_i, k_i > 0\ \forall i \in \{1,\dots,r\}$; $g>0$, $w=0$, $g_{j'}>0$, $w_{j'}>0$, $\forall j' \in \{1,\dots,R\}$; $p_{i'}$, $(i'=1,\dots,R)$ is an arbitrary positive integer and the coefficients $A\,[q_1,\alpha_1;\dots;q_R,\alpha_R]$ are arbitrary constant, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r} h_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1,\ Re\left(\sigma + \sum_{i=1}^{r} k_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, T_i>0,\ |\arg z_i| < 1/2$$

$T_i\pi$, $i=1,\dots r$, $j=1,\dots,u^{(i)}$, $\alpha>-1$, $\beta>-1$, $M=N$ ($M=N+1$ and $|y|<1$).

Proof. To establish (2.1), express the generalized prolate spheroidal wave function as given in (1.2), the generalized hypergeometric function as infinite series (Rainville [8], p.73. eqn. (2)), and the general polynomials with the help of equation (1.3), change the order of integration and summation (which is easily seen to be justified due to the absolute convergence of the integral and sums involved in the process) and then evaluate the inner integral by using a result of Gupta ([6], p.31, eqn. (2.2.1)), we arrive at the right hand side of (2.1).

3. Expansion Formula.

$$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\ {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix} ; y(1-x)^{g}(1+x)^{w}\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$(1+x)^{w_1}, \ldots y_R(1-x)^{g_R}(1+x)^{w_R}]\, H\, [z_1(1-x)^{h_1}(1+x)^{k_1}, \ldots z_r(1-x)^{h_r}(1+x)^{k_r}]dx$

$$= \sum_{l=0}^{\infty} E_l \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}\, \phi_n^{\alpha,\beta}(s,x),\, \phi_l^{\alpha,\beta}(s,x)\, dx \qquad \ldots (3.3)$$

Using the orthogonality property for the generalized prolate spheroidal wave functions (1.5) on the right-hand side and the result (2.1) on the left hand side of (3.3) we obtain

$$E_l = \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq} L(y_1,\ldots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q y^q}{q!\,(f_N)_q}$$

$$\sum_{m}^{l+p} \frac{(-l-p)_m(\alpha+\beta+l+p+1)_m}{m!\,(\alpha+1)_m} H^{0,\lambda+2:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho-gq-g_1\alpha_1-\ldots-g_R\alpha_R: h_1,\ldots,h_r],\,[-\sigma-wq-w_1\alpha_1-\ldots-w_R\alpha_R: k_1,\ldots,k_r],\\ [(c):\psi',\ldots,\psi^{(r)}]:[-1-m-\rho-\sigma-gq-wq-(g_1+w_1)\alpha_1-\ldots-(g_R+w_R)\alpha_R: h_1+k_1,\ldots,h_r+k_r]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\; z_1 2^{h_1+k_1},\ldots,z_r 2^{h_r+k_r}\right) / \{N_{n,l}^{\alpha,\beta}\delta_{n,l}\} \qquad (3.4)$$

with the help of (3.3) and (3.4) in view of (1.6), the expansion formula (3.1) is obtained.

4. Special Cases.

(i) Letting $\lambda = A$, $u^{(i)}=1$, $v^{(i)}=B^{(i)}$, $D^{(i)}=D^{(i)}+1$ $\forall i=1,\ldots,r$ in (2.1), the multivariable H-function transforrms to the generalized Lauricella function of several complex variables ([13],p. 454).

$$\int_{-1}^{1} (1-x)^{\rho}(1+x)^{\sigma}\, \phi_n^{\alpha,\beta}(s,x)\; {}_MF_N\left[\begin{matrix}e_M\\ f_N\end{matrix}; y(1-x)^{g}(1+x)^{w}\right] S^{p_1,\ldots,p_R}_{q_1,\ldots,q_R}[y_1(1-x)^{g_1}(1+x)^{w_1}$$

$$,\ldots y_R(1-x)^{g_R}(1+x)^{w_R}]\, F^{A:B';\ldots;B^{(r)}}_{C:D';\ldots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\ldots,\theta^{(r)}]:[1-(b'):\varphi'];\ldots;[1-(b^{(r)}):\varphi^{(r)}];\\ [1-(c):\psi',\ldots,\psi^{(r)}]:[1-(d'):\delta'];\ldots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix}\right.$$

$$\left. -z_1(1-x)^{h_1}(1+x)^{k_1},\ldots z_r(1-x)^{h_r}(1+x)^{k_r}\right]dx$$

$$= \sum_{p,q=0}^{\infty} 2^{\rho+\sigma+1+gq+wq} L(y_1,\ldots,y_R)\, R_{p,n}^{\alpha,\beta}(s)\, \frac{(e_M)_q y^q}{q!\,(f_N)_q} \sum_{m=0}^{n+p}$$

$$\frac{(-n-p)_m(\alpha+\beta+n+p+1)_m}{m!\,(\alpha+1)_m}\; \frac{\Gamma(1+m+\rho+gq+g_1\alpha_1+\ldots+g_R\alpha_R)\Gamma(1+\sigma+wq+w_1\alpha_1+\ldots+w_R\alpha_R)}{\Gamma(2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R)}$$

$$F^{A+2:B';\ldots;B^{(r)}}_{C+1:D';\ldots;D^{(r)}}\left(\begin{matrix}[1+m+\rho+gq+g_1\alpha_1+\ldots+g_R\alpha_R: h_1,\ldots,h_r],\,[1+\sigma+wq+w_1\alpha_1\\ [1\text{-}(c):\psi',\ldots,\psi^{(r)}]:[2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\ldots+(g_R+w_R)\alpha_R\end{matrix}\right.$$

$$\left.\begin{matrix}+\ldots+w_R\alpha_R: k_1,\ldots,k_r],[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];\\ :h_1+k_1,\ldots,h_r+k_r]:[1\text{-}(d'):\delta'];\ldots;[1\text{-}(d^{(r)}):\delta^{(r)}];\end{matrix}\; -z_1 2^{h_1+k_1},\ldots,-z_r 2^{h_r+k_r}\right) \qquad \ldots (4.1)$$

valid under the same conditions as obtainable from (2.1)

(ii) If we take $R = 1$ in (2.1) we get an integral involving the product of general class of polynomials, the generalized prolate spheroidal wave function, generalized hypergeometric function and a multivariable H-funciton

(iii) Letting $\lambda = A$, $u^{(i)}=1$, $v^{(i)}=B^{(i)}$, $D^{(i)}=D^{(i)}+1$ $\forall i=1,\dots,r$ in (3.1), we get an expansion formula for the product of hypergeometric function, generalized polynomials and the generalized Lauricella function of several complex variables ([13],p. 454)

$$(1-x)^{\rho-\alpha}(1+x)^{\sigma-\beta}\ {}_MF_N\left[\begin{matrix} e_M \\ f_N \end{matrix} ; y(1-x)^g(1+x)^w\right] S^{p_1,\dots,p_R}_{q_1,\dots,q_R}[y_1(1-x)^{g_1}(1+x)^{w_1},\dots, y_R(1-x)^{g_R}(1+x)^{w_R}]\ F^{A:B';\dots;B^{(r)}}_{C:D';\dots;D^{(r)}}\left(\begin{matrix}[1-(a):\theta',\dots,\theta^{(r)}]:[1-(b'):\varphi'];\dots;[1-(b^{(r)}):\varphi^{(r)}]; \\ [1-(c):\psi',\dots,\psi^{(r)}]:[1-(d'):\delta'];\dots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix} -z_1(1-x)^{h_1}(1+x)^{k_1},\dots,-z_r(1-x)^{h_r}(1+x)^{k_r}\right)$$

$$= \sum_{t,p,q=0}^{\infty} 2^{\rho+\sigma-\alpha-\beta+gq+wq} L(y_1,\dots,y_R)\, R^{\alpha,\beta}_{p,t}(s)\, \frac{(e_M)_q\, y^q}{q!\,(f_N)_q} \sum_{m=0}^{n+p}$$

$$\frac{(-t-p)_m(\alpha+\beta+t+p+1)_m}{m!\,(\alpha+1)_m}\ \frac{\Gamma(1+m+\rho+gq+g_1\alpha_1+\dots+g_R\alpha_R)\Gamma(1+\sigma+wq+w_1\alpha_1+\dots+w_R\alpha_R)}{\Gamma(2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\dots+(g_R+w_R)\alpha_R)}$$

$$F^{A+2:B';\dots;B^{(r)}}_{C+1:D';\dots;D^{(r)}}\left(\begin{matrix}[1+m+\rho+gq+g_1\alpha_1+\dots+g_R\alpha_R: h_1,\dots,h_r],[1+\sigma+wq+w_1\alpha_1+\dots+w_R\alpha_R: k_1,\dots,k_r],[(a):\theta',\dots,\theta^{(r)}]:[(b'):\varphi'];\dots;[b^{(r)}:\varphi^{(r)}]; \\ [1-(c):\psi',\dots,\psi^{(r)}]:[2+m+\rho+\sigma+gq+wq+(g_1+w_1)\alpha_1+\dots+(g_R+w_R)\alpha_R : h_1+k_1,\dots,h_r+k_r]:[1-(d'):\delta'];\dots;[1-(d^{(r)}):\delta^{(r)}];\end{matrix} -z_12^{h_1+k_1},\dots,-z_r2^{h_r+k_r}\right)$$

$$\left\{\phi_t^{\alpha,\beta}(s,x)/(R^{\alpha,\beta}_{p,t}(s))^2\, \frac{\Gamma(t+\alpha+p+1)\Gamma(t+p+\beta+1)}{(2t+2p+\alpha+\beta+1)\Gamma(t+p+1)\Gamma(t+p+\alpha+\beta+1)}\right\} \quad \dots (4.2)$$

valid under the same condition as obtainable from (3.1).

(iv) Taking $R = 1$ in (9), we get an expansion formula for the product of hypergeometric function, general class of polynomials and the multivariable H-function.

5. Known Results.

i. Letting $q_{j'}$ $(j' = 1,\dots,R) = 0$ in the equation (2.1) and (3.1), we get the results obtained by Gupta ([6], p.38, eqn. (2.3.2) and p. 44 eqn. (3.5.1)).

(ii) If we take $\alpha=\beta=n=0=s$, $w_1=0$, $g_1=1$ and $y=0$ in special case (ii) above, we get an integral similar to that obtained by Srivastava and Singh

with , $\xi=-1$ and $\eta=1$ ([16],p.166, eqn. (2.2)).

(iii) Letting $y=0$ in known result (i) we get an integral obtained by Gupta ([6], p. 50, eqn. (2.6.1)).

(iv) Setting $s=0$, $x=1-2\eta$ and using *Saalschutz's* theorem (Slater [9], p. 49) in known results (iii) above we get a result given by Srivastava and Panda ([15], p. 131, eqn.(2.2)).

(v) Taking $r=2$, $y=0$ and $\theta'=\theta''=\psi'=\psi''=\varphi'=\varphi''=\delta'=\delta''=1$ in eqn. (2.1) we arrive at a known result obtained by Mishra ([7],p. 158, eqn. 5.6.2).

(vi) Setting $y=0$, $A=0=C=\lambda$, $\upsilon''=B''=d''=0, u''=D''=\delta''=1$, $z_2\to 0$ in (2.1) and applying a known transformation formula Chaurasia ([1], P. 18, eqn. (1.5.4)), we arrive at another result given by Mishra ([7],p.157, eqn. (5.6.1)).

(vii) Setting $r=2$, $y=0$ and $\theta'=\theta''=\psi'=\psi''=\varphi'=\varphi''=\delta'=\delta''=1$ in eqn. (3.1) and using a known formula (Rainville [8],p.24, eqn. (2)), we obtain a result recorded by Mishra ([7],p.166, eqn. (5.7.5)).

(viii) Setting $r=2$, $A=0=C=$, $\upsilon''=B''=d''=0$, $u''=D''=\delta''=1$, $z_2\to 0$ in eqn. (3.1) and using a transformation formula (Chaurasia [1],p,.18, eqn. (1.5.4)), we get an expansion formula involving Fox's H-function.

(ix) When $y=0$, the known results (viii) reduces to another known result due to Mishra ([7], p.165, eqn. (5.7.4)).

Several other interesting special cases of our results can be deduced.

Acknowledgement

The authors are thankful to Professor H.M. Srivastav, University of Victoria, Canada, for valuable suggestions in the preparation of this paper.

REFERENCES

[1] V.B.L. Chaurasia, *Investigations in Integral Transforms and Special Functions*, Ph.D. thesis, University of Rajasthan, Jaipur, India, 1976.

[2] A. Erdélyi et al, *Tables of Integral Transforms*, Vol. II, Mcgraw Hill, New York, 1954.

[3] C. Flammer, *Spheroidal Wave Functions*. Stanford Univ. Press, Stanford, 1957.

[4] C. Fox, The G and H functions as symmetrical Fourier kernels, *Tran. Amer.Math. Soc.* **98** (1961), 395-429.

[5] R.K. Gupta, Generalized prolate spheroidal wave function, *Proc. Indian Acad. Sci. Sect.* A **85** (1977), 104-114.

[6] V.G. Gupta, *A Study of multi-valued function and their applications*, Ph.D. Thesis, University of Rajasthan, Jaipur, India, 1991

[7] S.D. Mishra, *Some Problem in Transform Calculus*, Ph.D. thesis, University of Rajasthan, Jaipur, India, 1977.

[8] E.D. Rainville, *Special Functions*, Macmillan, New York, 1960.

[9] L.J. Slater, *Generalized Hypergeometric Functions*, Cambridge Univ. Prss, Cambridge, 1966.

[10] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H- function of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[11] H.M. Srivastava, A Multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific. J. Math.* **117** (1985), 183-191.

[12] H.M. Srivastava, A Contour integral involving Fox's *H*-function, *Indian J. Math.* **14** (1972), 1-6.

[13] H.M. Srivastava and M.C. Daoust, Certain generalized, Neumann equations associated with the Kampé de Fériet function, *Nederl, Akad. Wetensch. Proc. Ser.* A **72** *Indag. Math.,* **31**, (1969), 449-457.

[14] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[15] H.M. Srivastava and R.Panda, Expansion theorems for the *H*-Function of several complex variables, *J. Reine Angew. Math.* **288** (1976), 129-145.

[16] H.M. Srivastava and N.P. Singh, The Integration of certain product of the multivariable *H*-function with general class of polynomials, *Rend. Circ. Mat. Palermo* (2). **32,** (1983), 157-187.

Jñānābha, Vol. 30, 2000

THE INTEGRATION OF CERTAIN PRODUCTS OF THE MULTIVARIABLE *H*-FUNCTION WITH GENERAL POLYNOMIALS AND EXTENDED JACOBI POLYNOMIALS

By

V.G. Gupta

Department of Mathematics

University of Rajasthan, Jaipur-302 004, Rajasthan

and

Suman Jain

S.S. Jain Subodh Postgraduate College, Jaipur-302 004, Rajasthan

(Received : August 30, 2000)

ABSTRACT

In this paper the authors present four integral formulas for the *H*-function of several complex variables, which was introduced and studied by H.M. Srivastava and R. Panda ([16, 17]; see also [11]). Each of these integral formulas involves a product of the multivariable *H*-function, extended Jacobi polynomials and general polynomials with essentially arbitrary coefficients, which were considered else where by H.M. Srivastava [13]. By assigning suitable special values to these coefficients, the main results (contained in the theorem 1 and 2) can be reduced to integrals' involving the classical orthogonal polynomials including, for example, Hermite, Jacobi, [and, of course, Gegenbauer (or ultraspherical), Legendre, and Tchebycheff], and Laguerre polynomials, the Bessel polynomials considered by H.L. Krall and O. Frink [8], and such other classes of generalized hypergeometric polynomials as those studied earlier by F. Brafman [2] an by H.W. Gould and A.T. Hopper [7]. On the other hand, the multivariables *H*-function occuring in each of our main results can be reduced, under various special cases, to such simpler functions as the generalized Lauricella hypergeometric functions of several complex variables [due to H.M. Srivastava and M.C. Daoust (cf. [14] and [15])], which indeed include a great many of the useful functions (or the products of several such functions) of hypergeometric type (in one or more variables) as their particular cases (see, e.g., [1],[9]). We record here only two special cases of our main integrals, the first one involves the extended Jacobi

Polynomials, general polynomials and multivariable H-function, and the other one involves the extended Jacobi polynomials, general polynomials and Lauricella function of several complex variables. Out of several known results which follow as special cases of our integrals we refer here only to the results of Srivastava and Singh [18], Srivastava and Panda [17] and Sharma [10].

1 Introduction. The H-Function of several complex variables defined by Srivastava and Panda ([16], and [17]) by means of the multiple Mellin-Barnes type integral (see also [11], p. 251, eqn. (C–1) et. seq.)

$$H\begin{bmatrix} z_1 \\ \vdots \\ z_r \end{bmatrix} = H^{0,\lambda:(u',v');\ldots;(u^{(r)},v^{(r)})}_{A,\,C:(B',D');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(c):\psi',\ldots,\psi^{(r)}]:[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix} z_1,\ldots,z_r\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r} U_1(\xi_1)\ldots U_r(\xi_r)\, V(\xi_1,\ldots,\xi_r)\, z_1^{\xi_1}\ldots x_R^{\xi_r}\, d\xi_1\ldots d\xi_r,\ \omega=\sqrt{-1} \qquad \ldots(1.1)$$

where

$$U_i(\xi_i) = \frac{\prod_{j=1}^{u^{(i)}}\Gamma[d_j^{(i)}-\delta_j^{(i)}\xi_i]\ \prod_{j=1}^{v^{(i)}}\Gamma[1-b_j^{(i)}+\phi_j^{(i)}\xi_i]}{\prod_{j=u^{(i)}+1}^{D^{(i)}}\Gamma(1-d_j^{(i)}+\delta_j^{(i)}\xi_j)\ \prod_{j=v^{(i)}+1}^{B^{(i)}}\Gamma(b_j^{(i)}-\phi_j^{(i)}\xi_i)},\qquad \forall i\in\{1,\ldots,r\} \qquad \ldots(1.2)$$

$$V(\xi_1,\ldots,\xi_r) = \frac{\prod_{j=0}^{\lambda}[1-a_j+\sum_{i=1}^{r}\theta_j^{(i)}\xi_i]}{\prod_{j=\lambda+1}^{A}[a_j-\sum_{i=1}^{r}\theta_j^{(i)}\xi_i]\ \prod_{j=1}^{C}\Gamma[1-c_j+\sum_{i=1}^{r}\psi_j^{(i)}\xi_i]} \qquad \ldots(1.3)$$

The multiple integral in (1.1) converges absolutely, if

$$T_i > 0 \text{ and } |\arg z_i| < T_i\pi/2 \quad \forall i\,\{1,\ldots r\} \qquad \ldots(1.4)$$

where

$$T_i = \sum_{j=1}^{\lambda}\theta_j^{(i)} - \sum_{j=\lambda+1}^{A}\theta_j^{(i)} + \sum_{j=1}^{v^{(i)}}\phi_j^{(i)} - \sum_{j=v^{(i)}+1}^{B^{(i)}}\phi_j^{(i)} - \sum_{j=1}^{C}\psi_j^{(i)} + \sum_{j=1}^{u^{(i)}}\delta_j^{(i)} - \sum_{j=u^{(i)}+1}^{D^{(i)}}\delta_j^{(i)} > 0,\ \forall i\,\{1,\ldots r\} \qquad \ldots(1.5)$$

These conditions are assumed to be satisfied by the various H-function of several variables occuring in this paper.

The general polynomials (multivariable) defined by Srivastava [12] represented in the following manner :

$$S^{p_1,\ldots,p_S}_{q_1,\ldots,q_S}\begin{bmatrix} x_1 \\ \vdots \\ x_S \end{bmatrix} = S\,[x_1,\ldots,x_S]$$

$$= \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_S=0}^{[q_S/p_S]} \frac{(-q_1)_{p_1k_1}}{k_1!} \cdots \frac{(-q_s)_{p_sk_s}}{k_s!} . A\,[q_1,k_1; \ldots ;q_s,k_s]$$

$$x_1^{k_1} \ldots x_s^{k_s} \qquad \ldots (1.6)$$

where $q_{i'} = 0,1,2,\ldots; p_{i'} \neq 0$ $(i' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1; \ldots ;q_s,k_s]$ are arbitrary constants real or complex.

If we take $s = 1$ in the equation (1.6) and denote $A\,[q,k]$ thus obtained by $A_{q,k}$, we arrive at the general class of polynomials $S_q^p(x)$ intorducted by Srivastava ([13], p.1, eqn. (1)).

Fujiwara [6] defined a generalized classical polynomials $R_n(x)$ on an interval (p,q) by means of the following Rodrigues' type formula:

$$R_n(x) = \frac{(-1)^n}{n!\,W(x)} \frac{d^n}{dx^n} [W(x)\{\upsilon(x)\}^n] \qquad \ldots(1.7)$$

where $\upsilon(x) = \mu\,(x-p)(q-x)$

$$W(x) = \frac{(x-p)^{\beta}(q-x)^{\alpha}}{(q-p)^{\alpha+\beta+1}\,B\,(\alpha+1, \beta+1)}\ \alpha > -1,\ \beta > -1 \qquad \ldots (1.8)$$

The class of polynomials $R_n(x)$ provides a unification of the classical orthogonal polynomials such as Jacobi, Laguerre and Hermite etc. The polynomials denoted by $F_n(\beta;\alpha;x)$ are called extended Jacobi Polynomials.

Thakare [20] studied the Fujiwara's polynomials extensively and obtained the following hypergeometric form of $F_n(\beta;\alpha;x)$.

$$F_n(\beta;\alpha;x) = \frac{(-\mu)^n\,(1+\beta)_n}{n!} (q-x)^n\ {}_2F_1\left(\begin{matrix}-n,-n-\alpha \\ 1+\beta\end{matrix} ; \frac{p-x}{q-x}\right),\ p < x < q \quad \ldots(1.9)$$

When $p = -1$, $q = 1$ and $\mu = 1/2$ in (1.9), $F_n(\beta;\alpha;x)$ reduces to the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$.

Lemma. If $Re\,(\rho) > -1$, $Re\,(\sigma) > -1$ and $p \neq q$, then

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\,dx = \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!\,\Gamma\,(\rho+\sigma+n+2)} (q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n} \frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(\rho+m+1)\,\Gamma\,(\sigma+n+1-m)}{m!\,(1+\beta+m)} \qquad \ldots (1.10)$$

Proof. The above lemma follows at once by applying the definition (1.9) in conjunction with the following form of the well known Eulerian integral for the Beta function

$$\int_p^q (x-p)^{\rho-1}(q-x)^{\sigma-1}\,dx = (q-p)^{\rho+\sigma+n-1} \frac{\Gamma(\rho)\,\Gamma(\sigma)}{\Gamma\,(\rho+\sigma)}, p \neq q \qquad \ldots (1.11)$$

where for convergence, $min\,\{Re\,(\rho), Re(\sigma)\} > 0$

2. The main integral formulas. Our main reults of the present paper are the integral formulas contained in the following theorems:

Theorem 1. With t_i defined by (1.5) let $|arg\, z_i| < T_i\pi/2 \;\; \forall i \; \{1,...r\}$, where each of the equali Ties holds for suitably restricted values of the complex variables $z_1,...z_r$. Also let the polynomials $F_n(\beta;\alpha;x)$ be defined by (1.9) for every postive integer n.

Then

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x) \; S^{p_1,...,p_s}_{q_1,...,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s} \end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,...,y_s)$$

$$H^{0,\lambda+2:(u',v');...;(u^{(r)},v^{(r)})}_{A+2,C+1:(B',D');...;(B^{(r)},D^{(r)})}\left(\begin{matrix}[-m-\rho-g_1k_1-...-g_sk_s:\rho_1,...,\rho_r], \\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-...-(g_s+w_s)g_s:\rho_1+\sigma_1,...,\rho_r+\sigma_r]\end{matrix}\right.$$

$$[m-\sigma-n-w_1k_1-...-w_sk_s:\sigma_1,...,\sigma_r],[(a):\theta',...,\theta^{(r)}]:[(b'):\varphi'];...;[b^{(r)}:\varphi^{(r)}];$$
$$[(c):\psi',...,\psi^{(r)}]:[(d'):\delta'];[(d''):\delta''];...;[d^{(r)}:\delta^{(r)}];$$
$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},..., z_r(q-p)^{\sigma_r}\right) \qquad ...(2.1)$$

where

$$L(y_1,...,y_s) = \sum_{k_1=0}^{[q_1/p_1]} \cdots \sum_{k_s=0}^{[q_s/p_s]} \prod_{j'=1}^{s}\left[\frac{(-q_{j'})p_{j'}k_{j'}}{k_{j'}!} y_{j'}^{k_{j'}}(q-p)^{(g_{j'}+w_{j'})k_{j'}}\right]$$

$$A[q_1,k_1;...;q_s,k_s] \qquad ...(2.2)$$

provided that $p \neq q$, $\rho_i, \sigma_i > 0 \; \forall i \in \{1,...,s\}$; $g_j, w_j > 0$, $\forall j \in \{1,...,s\}$; $p_{i'}$, $(i' = 1,...,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;...;q_s,k_s]$ are arbitrary constant, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r} \rho_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, \; Re\left(\sigma + \sum_{i=1}^{r} \sigma_i \frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1, \qquad ...(2.3)$$

and of course $T_i > 0$, $|arg\, z_i| < T_i\pi/2 \;\; i = 1,...r$, $j = 1,...,u^{(r)}$.

Theorem 2. Under hypothesis preceding the assertion (2.1)of theorem 1,

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x) \; S^{p_1,...,p_s}_{q_1,...,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s} \end{bmatrix} H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2(q-x)^{\sigma_2} \\ \vdots \\ z_r(q-x)^{\sigma_r} \end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n\,!} (q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m (-n)_m (-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y_s)$$

$$H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',v'');\ldots;\,(u^{(r)},v^{(r)})}_{A+1,C+1:\,(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})} \left(\begin{matrix} [-m-\sigma-w_1k_1-\ldots-w_sk_s:\sigma_1,\ldots,\sigma_r], \\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\ldots-(g_s+w_s)\,k_s: \end{matrix} \right.$$

$$[(a):\theta',\ldots,\theta^{(r)}]:[(b'):\varphi'];[(b''):\varphi''];\ldots;[b^{(r)}:\varphi^{(r)}];$$

$$\sigma_1-\epsilon,\sigma_2,\ldots,\sigma_r],[(c):\psi',\ldots,\psi^{(r)}]:[\rho+1+m+g_1k_1+\ldots+g_sk_s:\epsilon],[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];$$

$$\left. z_1(q-p)^{\sigma_1-\epsilon}, z_2(q-p)^{\sigma_2},\ldots,z_r(q-p)^{\sigma_r} \right) \quad \ldots(2.4)$$

The Integral (2.4) holds provided that $p \neq q$, $\sigma_1 > \epsilon > 0$, $\sigma_j > 0\ j = 2,\ldots r$; $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\ldots,s\}$; $p_{j'}\ (i' = 1,\ldots,s)$ are arbitrary positive integers and the coefficients $A\,[q_1,k_1;\ \ldots\ ;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho-\epsilon\frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma+\sum_{i=1}^{r}\sigma_i\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > -1 \quad \ldots(2.5)$$

and of course $T_i > 0$, $|arg\ z_i| < T_i\pi/2$, $i = 1,\ldots r$, $j = 1,\ldots,u^{(r)}$.

$$\int_p^q (x-p)^\rho (q-x)^\sigma F_n(\beta;\alpha;x)\ S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}\begin{bmatrix} y_1(x-p)^{g_1}(q-x)^{w_1} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma} \\ z_2(x-p)^{\rho_2} \\ \vdots \\ z_r(x-p)^{\rho_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n \Gamma(1+\beta+n)}{n\,!} (q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m (-n)_m (-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y^s)$$

$$H^{0,\lambda+2\,:\,(u',+1,v');\,(u'',v'');\ldots;\,(u^{(r)},v^{(r)})}_{A+1,C+1:\,(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})} \left(\begin{matrix} [-\rho-m-g_1k_1-\ldots-g_sk_s:\rho_1,\ldots,\rho_r], \\ [-1-n-\rho-\sigma-(g_1+w_1)k_1-\ldots-(g_s+w_s)k_s: \end{matrix} \right.$$

$$[(a):\theta',\ldots,\theta^{(r)}]:$$

$$\rho_1-\gamma,\rho_2,\ldots,\rho_r],[(c):\psi_1,\ldots,\psi^{(r)}]:[1+\sigma+n-m+w_1k_1+\ldots+w_sk_s:\gamma];$$

$$[(b'):\varphi'];[(b''):\varphi''];\ldots;[b^{(r)}:\varphi^{(r)}]; \quad \left. z_1(q-p)^{\rho_1+\sigma_1},\ldots,z_r(q-p)^{\rho_r+\sigma_r}\right) \quad \ldots(2.6)$$

$$[(d'):\delta'];[(d''):\delta''];\ldots;[d^{(r)}:\delta^{(r)}];$$

The Integral (2.6) holds provided that $p \neq q$, $\rho_1 > \gamma > 0$, $\rho_j > 0\ j = 2,\ldots r$; $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\ldots,s\}$; $p_{j'}\ (j' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A\,[q_1,k_1;\ \ldots\ ;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho + \sum_{i=1}^{r} \rho_i \frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma - \gamma \frac{d'_j}{\delta'_j}\right) > -1 \qquad \ldots(2.7)$$

and of course $T_i > 0, |arg\, z_i| < T_i \pi/2$, $i = 1, \ldots r, j = 1,\ldots,u^{(r)}$.

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}\begin{bmatrix} y_1(x-p)^{g_i}(q-x)^{w_i} \\ \vdots \\ y_s(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix} H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma} \\ z_2 \\ \vdots \\ z_r\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1} \sum_{m=0}^{n} \frac{(-1)^m(-n)_m(-n-\alpha)_m}{m!\,\Gamma(1+\beta+m)} L(y_1,\ldots,y_s)$$

$$H^{0,\lambda:(u'+2,v');(u'',v'');\ldots;(u^{(r)},v^{(r)})}_{A,C:(B'+1,D'+2);(B'',D'');\ldots;(B^{(r)},D^{(r)})}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]: \\ [(c):\psi 1,\ldots,\psi^{(r)}]:[(d'):\delta'];\end{matrix}\right.$$

$$[\rho+\sigma+n+2+(g_1+w_1)k_1+\ldots+(g_s+w_s)k_s:\gamma+\epsilon];[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];$$
$$[\rho+m+1+g_1k_1+\ldots g_sk_s:\epsilon],[\sigma+n-m+1+w_1k_1+\ldots+w_sk_s:\gamma];[(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];$$
$$\left. z_1(q-p)^{-\gamma-\epsilon}, z_2\ldots, z_r\right) \qquad \ldots(2.8)$$

The Integral (2.8) holds provided that $p \neq q$, $\gamma > 0$, $\epsilon > 0$, $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,\ldots,s\}$; $p_{j'}$ $(j' = 1,\ldots,s)$ is an arbitrary positive integer and the coefficients $A[q_1,k_1;\ldots;q_s,k_s]$ are arbitrary constants, real or complex.

$$Re\left(\rho - \epsilon\frac{d_j'}{\delta_j'}\right) > -1,\ Re\left(\sigma - \gamma\frac{d'_j}{\delta'_j}\right) > -1 \qquad \ldots(2.9)$$

and of course $T_i > 0, |arg\, z_i| < T_i\pi/2$, $i = 1, \ldots r, j = 1,\ldots,u^{(r)}$,
where $L(y_1,\ldots,y_s)$ in (2.4), (2.6) and (2.8) is same as given in (2.2)

Proof. To establish (2.1) express the generalized polynomials with the help of equation (1.6) and the multivariable H-function in terms of Mellin-Barnes type Contour integral by virtue of (1.1), interchanging the order of intergration and summation (which is easily seen to be permissible under the conditions stated) and then evaluate the x-integral with the help of (1.10). On interpreting the result thus obtained by virtue of (1.1), we arrive at the right hand side of (2.1).

The remaining integrals (2.4), (2.6) and (2.8) can be evaluated in a similar manner.

3. Special Cases:

(i) Letting $s = 1$, in eqn. (2.1), (2.4), (2.6) and (2.8), we get interesting integrals involving extended Jacobi polynomials, general class of polynomials and multivariable H-function as follows :

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y(x-p)^g (q-x)^w]\ H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1}\sum_{m=0}^{n}\sum_{k=0}^{[q/p]}\frac{(-1)^m(-n)_m(-n-\alpha)_m(-q)_{pk}}{m!\,\Gamma(1+\beta+m)\,k!}.$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^k\ \ H^{0,\lambda+2\,:\,(u',\,v');\,\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+2,C+1:\,(B',D');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-m-\rho-gk:\rho_1,\ldots,\rho_r],\ [m-\sigma-n-wk:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]\\ [1-n-\rho-\sigma-(g+w)k:\rho_1+\sigma_1,\ldots,\rho_r+\sigma_r]\ [(c):\psi',\ldots,\psi^{(r)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}:[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1(q-p)^{\rho_1+\sigma_1},\ldots,z_r(q-p)^{\rho_r+\sigma_r}\right) \qquad \ldots(3.1)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y(x-p)^g (q-x)^w]\ H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{\sigma_1} \\ z_2(q-x)^{\sigma_2} \\ \vdots \\ z_r(q-x)^{\sigma_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1}\sum_{m=0}^{n}\sum_{k=0}^{[q/p]}\frac{(-1)^m(-n)_m(-n-\alpha)_m(-q)_{pk}}{m!\,\Gamma(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}]^k\ \ H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',\,v'');\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+1,C+1:\,(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[m-\sigma-n-wk:\sigma_1,\ldots,\sigma_r],\ [(a):\theta',\ldots,\theta^{(r)}]:\\ [-1-n-\rho-\sigma-(g+w)k:\sigma_1-\epsilon,\sigma_2,\ldots,\sigma_r],[(c):\psi',\ldots,\psi^{(r)}]:\ [\rho+1+m+gk:\epsilon];\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}];\\ [(d'):\delta'];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\ z_1(q-p)^{\sigma_1-\epsilon}\,z_2(q-p)^{\sigma_2},\ldots,z_r(q-p)^{\sigma_r}\right) \qquad \ldots(3.2)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma} F_n(\beta;\alpha;x)\ S_q^p[\,y(x-p)^g (q-x)^w]\ H\begin{bmatrix} z_1(x-p)^{\rho_1}(q-x)^{-\gamma} \\ z_2(x-p)^{\rho_2} \\ \vdots \\ z_r(x-p)^{\rho_r}\end{bmatrix} dx$$

$$= \frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n!}(q-p)^{\rho+\sigma+n+1}\sum_{m=0}^{n}\sum_{k=0}^{[q/p]}\frac{(-1)^m(-n)_m(-n-\alpha)_m(-q)_{pk}}{m!\,\Gamma(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}\,]^k \quad H^{0,\lambda+1\,:\,(u',+1,v');\,(u'',\,v'');\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A+1,C+1:\;(B',D'+1);(B'',D'');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[-\rho-m-gk:\,\rho_1,\,\ldots\,,\rho_r],\;\; [(a):\theta',\ldots,\theta^{(r)}]: \\ [-1-n-\rho-\sigma-(g+w)k:\rho_1-\gamma,\rho_2,\ldots,\rho_r],[(c):\psi',\ldots,\psi^{(r)}]:[1+\sigma+n-m+wk:\gamma],\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\,\delta'\,];\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\; z_1(q-p)^{\rho_1-\gamma},z_2(q-p)^{\rho_2},\ldots,\,z_r(q-p)^{\rho_r}\right) \qquad \ldots(3.3)$$

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma}\,F_n(\beta;\alpha;x)\;\;S_q^p\,[\,y\,(x-p)^g\,(q-x)^w\,]\;H\begin{bmatrix} z_1(x-p)^{-\epsilon}(q-x)^{-\gamma} \\ z_2 \\ \vdots \\ z_r\end{bmatrix}dx$$

$$=\frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1}\;\sum_{m=0}^{n}\sum_{k=0}^{[q/p]}\;\frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m\,(-q)_{pk}}{m!\,\Gamma\,(1+\beta+m)\,k!}$$

$$A_{q,k}\,[y\,(q-p)^{g+w}\,]^k\quad H^{0,\lambda\,:\,(u'+2,v');\,(u'',\,v'');\ldots\,;\,(u^{(r)},\,v^{(r)})}_{A,C:\;(B'+1,D'+2);(B'',D'');\ldots;(B^{(r)},D^{(r)})}$$

$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(r)}]:\,[\rho+\sigma+n+2+\,(g+w)k:\,\gamma+\epsilon]; \\ [(c):\,\psi',\ldots,\psi^{(r)}]:\,[\rho+m+1+gk:\,\epsilon],\,[\sigma+n-m+1+wk:\gamma]\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\varphi'];\ldots;[b^{(r)}:\varphi^{(r)}]; \\ [(d'):\,\delta'\,];\,\ldots;[d^{(r)}:\delta^{(r)}];\end{matrix}\qquad z_1(q-p)^{-\gamma-\epsilon}\,,\,z_2,\ldots,\,z_r\right) \qquad \ldots(3.4)$$

which are valid under essentially the same conditions as those of their parent formulas (2.1), (2.4), (2.6), and (2.8), respectively.

(ii) Giving suitable values to parameters and using a relation [17] in (2.1), (2.4), (2.6) and (2.8) we obtain the integrals involving generalized Lauricella function [15], but for the sake of brevity, they all are not presented here. (for example, putting $\lambda=A$, $u^{(i)}=1$, $v^{(i)}=B^{(i)}$, replecing $D^{(i)}$ by $D^{(i)}+1\;\forall i=1,\ldots,r$ in (2.1), we have

$$\int_p^q (x-p)^{\rho}(q-x)^{\sigma}\,F_n(\beta;\alpha;x)\;\;S^{p_1,\ldots,p_s}_{q_1,\ldots,q_s}\begin{bmatrix} y_1(x-p)^{g_1}\,(q-x)^{w_1} \\ \vdots \\ y_s\,(x-p)^{g_s}(q-x)^{w_s}\end{bmatrix}$$

$$F^{\,A:B',\ldots,B^{(r)}}_{\,C:D',\ldots,D^{(r)}}\begin{bmatrix} -z_1(x-p)^{\rho_1}(q-x)^{\sigma_1} \\ \vdots \\ -z_r(x-p)^{\rho_r}(q-x)^{\sigma_r}\end{bmatrix}dx\;=\;\frac{(-\mu)^n\,\Gamma(1+\beta+n)}{n\,!}(q-p)^{\rho+\sigma+n+1}$$

$$\sum_{m=0}^{n}\;\frac{(-1)^m\,(-n)_m\,(-n-\alpha)_m}{m!\,\Gamma\,(1+\beta+m)}\,L\,(y_1,\ldots,y_s)$$

$$\frac{\Gamma(1+\rho+m+g_1k_1+...g_sk_s)\Gamma(1+\sigma+n-m+w_1k_1+...+w_sk_s)}{\Gamma(2+n+\rho+\sigma+(g_1+w_1)k_1+...+(g_s+w_s)k_s)} F^{A+2:B';...;B^{(r)}}_{C+1:D';...;D^{(r)}}$$

$$\left(\begin{matrix}[1+m+\rho+g_1k_1+...+g_sk_s: \rho_1, ..., \rho_r], [1-m+\sigma+n+w_1k_1+...+w_sk_s:\sigma_1,...,\sigma_r], \\ [2+n+\rho+\sigma+(g_1+w_1)k_1+...+(g_s+w_s)k_s:\rho_1+\sigma_1,...,\rho_s+\sigma_r]:[1-(c):\psi',...,\psi^{(r)}]: \\ [1-(a):\theta',...,\theta^{(r)}]:[1-(b'):\varphi'];...;[1-b^{(r)}:\varphi^{(r)}]; \\ [1-(d'):\delta'];...;[1-d^{(r)}:\delta^{(r)}]; \qquad -z_1(q-p)^{\rho_1+\sigma_1},...,-z_r(q-p)^{\rho_s+\sigma_s}\end{matrix}\right) \quad ...(3.5)$$

valid within the domain of convergence of the resulting series where n is non-negative integer, $Re\ (\rho) > -1$, $Re\ (\sigma) > -1, \rho_i, \ \sigma_i > 0$ $\forall i = \{1,..,r\}$, $g_{j'}, w_{j'} > 0\ \forall j' \in \{1,...,s\}$

(iii) Taking $g = 0$, $w=1$, $\alpha=\beta=n=0$ the results in eqn. (3.1), (3.2), (3.3) and (3.4) reduce to the known results given by Srivastava and Singh ([18],p. 166, eqn. (2.2), (2.4), (2.6) and (2.8).

(iv) Letting $p=-1$, $q=1$, $\beta=\rho$, $\rho_1=\rho_2,...,\rho_r=0$, $\mu=1$ the result (3.1) reduces to a result obtained by Srivastava and Panda ([17]).

(v) Letting $\rho=\beta$, $q_j\ (j=1,...,s)=0$ in (2.1) we obtained a result givin by Sharma ([10],p. 57, eqn. (2.2.3)).

(vi) Letting $\sigma_1=\sigma_2=...=\sigma_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 56, eqn. (2.2.2)).

(vii) Letting $\rho_1=\rho_2=...=\rho_r=0$ in case (v) above we obtained a result given by Sharma ([10], p. 55, eqn. (2.2.1)

Several other special case of the integral established here can be obtained.

REFERENCES

[1] P., Appell, et kampé de fériet J., *Fonctions Hypergeometriques et Hyperspheriques; Polynomes d' Hermite*, Gauthier-Villars, Paris, 1926.

[2] F., Brafman, Some generating functions for Laguerre and Hermite polynomials, *Canad. J. Math.* **9** (1957), 180-187.

[3] S.P. Chiney and B.R., Bhonsle, Some resutlts involving extended Jacobi polynomials, *Rev. Univ. Nac. Tucuman*, **25** (1-2) (1975), 7-11.

[4] A. Erdélyi, et al., Higher Transcendental Functions, Vol. 1, McGraw-Hill Book Co. (1953).

[5] A., Erdélyi, et al., *Tables of Integral Transforms*, Vol. 2, McGraw- Hill Book Co. (1954).

[6] I., Fujiwara, A unified presentation of classical orthogonal polynomials, *Math. Japon*, **11** (1966), p. 133-148.

[7] H.W., Gould and A.T., Hopper, Operational formulas connected with two generalizations of Hermite polynomials, *Duke Math. J.* **29** (1962), 51-63.

[8] H.L. Krall and O. Frink, a new class of orthogonal polynomials: the Bessel polynomials, *Trans. Amer. Math. Soc.* **65** (1949), 100-115.

[9] G., Lauricella, sulle funzioni ipergeometriche a piu variabili, *Rend. Circ. Mat. Palemo* **7** (1893), 111-158.

[10] S.C. Sharma, Ph.D. Thesis, *University of Rajasthan, Jaipur, India, 1987.*

[11] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions fo One and Two Variables with Applications*, South Asian Publihers, New Delhi and Madras,

1982.

[12] H.M. Srivastava, A Multilinear generating function for the Konhauser sets of biorthogonal polynomials suggested by the Laguerre polynomials, *Paccific. J. Math.* 117 (1985) 183-191.

[13] H.M., Srivastava, A Contour integral involving Fox's *H*-function, *Indian J. Math.* 14 (1972), 1-6.

[14] H.M. Srivastava and M.C. Daoust, A note on the convegence of Kampé de Fériet's double hypergeometric series, *Math. Nechar.* 53 (1972), 151-159.

[15] H.M., Srivastava and M.C. Daoust, Certain generalized, Neumann equations associated with the kampé de fériet function, *Nederl, Akad. Wetensh. Proc. Ser. A-72 - Indag. Math.*, 31, 449-457 (1969).

[16] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* 283/284 (1976). 265-274.

[17] H.M. Srivastava and R. Panda, Expansion theorems for the *H*-function of several complex variables, *J. Reine Angew. Math.* 288 (1976), 129-145.

[18] H.M. Srivastava and N.P. Singh, The Integration of certain product of the multivariable *H*-function with general class of polynomials. *Rend. Circ. Mat. Palermo* (2). 32, (1983), 157-187.

[19] G., Szegö, Orthogonal Polynomials, *Amer. Math. Soc. Colloq. Publ.*, **Vol. XXIII**, Fourth edition, *Amer. Math. Soc., Providence. Rhode Island*, 1975.

[20] N.K. Thakare, *A Study of certain sets of orthogonal polynomials and their applications*, Ph.D. thesis, 1972, Shivaji University, Kolhapur.

Jñānābha, Vol. 30, 2000

A REMARK ON SASAKIAN RECURRENT SPACES OF THE FIRST KIND

By

A.K. Singh and K. L. Joshi

Department of Mathematics, H.N.B. Garhwal University,

Campus Tehri, Tehri Garhwal -249001, Uttar Pradesh, India

(Received : July 10, 2000)

1. Introduction. Let us consider an n-dimensional affinely connected sasakian space s_n with recurrent curvature R^i_{jkl} admitting an infinitesimal transformation :

$\overline{x}^i = x^i + v^i (x)\, \delta t,$

and we assume a basic condition $\underset{v}{\mathrm{L}} \lambda_m = 0$, where λ_m is a non-zero recurrence vector appeared in the equation

$$\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl},$$

$\underset{v}{\mathrm{L}}$ denotes the Lie-derivative with respect to v^i and ∇ denotes the operator of covariant differentiation with respect to the Riemannian connection. In this case, the space has been called a Sasakian recurrent space of the first kind and is denoted in brief by an S^*_n – space. Under a condition $\alpha = \lambda_m v^m \neq$ constant, we have

$$Akl \overset{def}{=\!=} \nabla_l \lambda_k - \nabla_k \lambda_l \neq 0 .$$

The present space with $\lambda \neq$ constant is called a Sasakian recurrent spaces of the first kind, or in brief an $-A\ S^*_n$–space.

Takano (1966) has studied the existence of an affine motion of recurrent form and give a remark there and it is read as follows

In this space satisfying $\underset{v}{\mathrm{L}} R^i_{jkl} = 0$, under a docomposition of curvature tensor of the form :

$\alpha R^i_{jkl} = Akl\, \nabla_j v^i$ in order to have $\underset{v}{\mathrm{L}} \Gamma^i_{jk} = 0$, we can suppose formally a parallel property of contravariant vector $\beta v^i - \alpha\, \eta^i$, but this supposition yields a contradiction.

However, generally speaking, the present space itself does not admit intrinsically such a vector field. In this short report, we shall prove this fact.

2. Some Formulae. The space with recurrent curvature is characterized by a basic condition

(2.1) $\nabla_m R^i_{jkl} = \lambda_m R^i_{jkl}$,

and the useful fundamental formulae are the first and second Bianchi's

identities given by

(2.2) $R^i_{klj} + R^i_{ljk} + R^i_{jkl} = 0,$

(2.3) $\lambda_m R^i_{jkl} + \lambda_k R^i_{jlm} + \lambda_l R^i_{jmk} = 0.$

An $A\ S^*_n$–space has not only (2.1), (2.2) and (2.3) but also the following formulae :

(2.4) $A_{kl}\, v^l = -\alpha.\alpha_k$,

(2.5) $\alpha_i v^i = 0$, *i.e.*, $\pounds_v \alpha = 0,$

and

(2.6) $A_{kl} = \lambda_k \alpha_l - \lambda_l \alpha_k.$

The well-known Ricci's identity in this space may be derived from (2.1) with ease and it is written as

(2.7) $A_{mn} R^i_{jkl} = R^\alpha_{jkl}\ R^i_{\alpha mn} - R^i_{\alpha kl} R^i_{jmn} - R^i_{j\alpha l} R^\alpha_{kmn} - R^i_{jk\alpha} R^i_{lmn}.$

3. Main Subject. Being $\alpha_k \neq 0$, we can introduce a vector η^k so as to satisfy

(3.1) $\alpha_k \eta^k = 1.$

Then, multiplying (2.7) by $\eta^m v^n$ side by side and summing over m and n, we get

(3.2) $\alpha R^i_{jkl} + R^\alpha_{jkl}\ A^i_\alpha\ - R^i_{\alpha kl} A^\alpha_j - R^i_{j\alpha l} A^\alpha_k - R^i_{jk\alpha} A^\alpha_l = 0$,

where we have used $A_{mn}\eta^m v^n = -\alpha$, derived from (2.4) and (3.1), or (2.5), (2.6) and (3.1) and put $A^i_j = R^i_{jkl}\eta^k v^l$.

Next, multiplying (2.3) by v^m and summing over m owing to $R^i_{jmk} = -R^i_{jkm}$, we find

(3.3) $\alpha R^i_{jkl} + \lambda_k R^i_{jlm} v^m - \lambda_l R^i_{jkm} v^m = 0,$

from which, putting $\beta = \lambda_m \eta^m$, we are able to have

(3.4) $\lambda_k A^i_j = (\beta v^l - \alpha\eta^l) R^i_{jkl}.$

Eliminating αR^i_{jkl} from (3.2) and (3.3), we have

$R^\alpha_{jkl} A^i_\alpha - R^i_{\alpha kl} A^\alpha_j = R^i_{j\alpha l}(A^\alpha_k - \lambda_k v^\alpha) - R^i_{j\alpha k}(A^\alpha_l - \lambda_l v^\alpha)$.

At this moment, if we multiply the last formula by $\beta v^l - \alpha\eta^l$ and sum over the index l, then by virtue of (3.4), it follows that

(3.5) $A^i_j\ \lambda_\alpha (A^\alpha_k - \lambda_k v^\alpha) = R^i_{j\alpha k}(A^\alpha_k - \lambda_l v^\alpha)(\beta v^l - \alpha\eta^l).$

Now, from $R^i_{jkl} = -R^i_{jlk}$ and (2.2), we see that $R^i_{jkl} - R_{kjl} = -R^i_{ljk}$, so from (3.4), we can derive the relation

$$\lambda_k A^i_j - \lambda_j A^i_k = -(\beta v^l - \alpha\eta^l) R^i_{ljk}.$$

Hereupon, let us assume the parallelism of $\beta v^i - \alpha\eta^i$, then we have immediately $(\beta v^l - \alpha\eta^l) R^i_{ljk} = 0$, so under the present assumption, it is concluded that

$\lambda_k A^i_j = \lambda_j A^i_k$.

Being $\lambda_k \neq 0$, from the last formula, we may put

(3.6) $A^i_k = \lambda_k A^i$,

where A^i indicates a suitable vector. Making use of (3.6) into the right hand side of (3.5) and taking care of $(\beta v^l - \alpha n^l)\,\lambda_l = 0$, we have $A^i_j\,(\mu_k - \alpha\lambda_k) = 0$, where we have defined μ_k by $A^i_k\lambda_i$,. It is easy to see that $A^i_j \neq 0$, and we have

(3.7) $\mu_k = \alpha\,\lambda_k$.

From (3.6), it follows that $\mu_k = \lambda_i A^i_k = \lambda_i A^i \lambda_k$ comparing the last result with (3.7), owing to, $\lambda_k \neq 0$ we get

(3.8) $\lambda_m A^m = \alpha$.

Substituting (3.6) into (3.2), we have

$$\alpha\, R^i_{jkl} - (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j - A^\alpha\,(\lambda_k R^i_{j\alpha l} - \lambda_l R^i_{jk\alpha}) = 0$$

or by virtue of (2.3), we get

$$\alpha\, R^i_{jkl} + A^\alpha\,\lambda_\alpha R^i_{jlk} = (\nabla_l \nabla_k - \nabla_k \nabla_l)\, A^i_j .$$

Making use of $R^i_{jkl} = -R^i_{jlk}$ and (3.8), the above equation becomes

$$\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l\, A^i_j = 0.$$

In this case, for a suitable vector δ_k, we can regard A^i_j to be a recurrent tensor given by

(3.9) $\nabla_k A^i_j = \delta_k\, A^i_j$.

Differentiating (3.4) covariantly and making use of parallelism of $\beta v^i - \alpha\eta^i$, we find

$$A^i_j \nabla_l \lambda_k + \lambda_k \delta_l A^i_j = \lambda_l \lambda_k A^i_j,$$

where we have used (2.1) and (3.9).

Being $A^i_j \neq 0$, from the last formula, we get

(3.10) $\nabla_l \lambda_k = \lambda_k \lambda_l - \lambda_k \delta_l$.

Now, remembering (3.6), (3.9) may be rewritten as

$$\lambda_j \nabla_k A^i + A^i A^i{}_k \lambda_j = \delta_k\, \lambda_j A^i.$$

Consequently, substituting (3.10) into the last relation, we get

(3.11) $\nabla_k A^i = (2\delta_k - \lambda_k) A^i$,

where we have neglected non-vanishing λ_j. Multiplying (3.11) by λ_i and summing over i, according to

$$\begin{aligned}\lambda_i \nabla_k A^i &= \nabla_k(\lambda_i A^i) - A^i \nabla_k \lambda_i \\ &= \alpha\,\alpha_k - A^i(\lambda_i\lambda_k - \lambda_i\delta_k) \\ &= \alpha\,\alpha_k - \alpha\lambda_k + \alpha\delta_k ,\end{aligned}$$

and making an appeal to (3.8) and (3.10), we have

$$\alpha\,\alpha_k - \alpha\lambda_k + \alpha\delta_k = \alpha\,(2\delta_k - \lambda_k),$$

or $\quad \alpha(\delta_k - \lambda_k) = 0,$

i.e. $\quad \delta_k = \lambda_k.$

In this way, (3.10) becomes

$$\nabla_l \lambda_k = \lambda_k \lambda_l - \lambda_k \alpha_l .$$

hence we can see that

$$Akl = \nabla_l \lambda_k - \nabla_k \lambda_l = -\lambda_k \alpha_l + \lambda_l \lambda_k .$$

Comparing the last result with (2.6), we have here $Akl = 0$ and so, we have

$$\alpha = c \text{ (constant)}.$$

This conclusing contradicts clearly with our preliminary assumption on $\alpha = \lambda_m \upsilon^m$, for the AS_n^*– space has been introduced orginally by basic condition $\underset{\upsilon}{\mathrm{L}} \lambda_m = 0$ and $\alpha \neq$ constant.

This completes the proof of the fact emphasized at the beginning of this report. Thus, we can state here a definite conclusion as follows :

The AS_n^*– space does not admit a parallel vector field defined by $\beta\upsilon^i - \alpha\eta^i$

4. Appendix. In the preceding section, under $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ we have supposed formally a condition (3.9). We shall show here the possibility of (3.9).

If $\nabla_l \nabla_k A^i_j - \nabla_k \nabla_l A^i_j = 0$ will be the case, the formula (3.2) is, simplified as

$$\alpha R^i_{jkl} = R^i_{j\alpha l} A^\alpha_k + R^i_{jk\alpha} A^\alpha_l .$$

Contracting $\eta^k \upsilon^l$ and using (3.6), we get

$$\alpha A^i_j = R^i_{j\alpha l} \upsilon^l (\eta^k \lambda_k) A^\alpha + R^i_{jk\alpha} \eta^k (\upsilon^l \lambda_l) A^\alpha,$$

say

$$\alpha A^i_j = R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) A^\alpha.$$

Differentiating the last formula covariantly and making use of (2.1) and the parallelism of $\beta\upsilon^l - \alpha\eta^l$, we have

$$\alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + R^i_{j\alpha l} (\beta \upsilon^l - \alpha \eta^l) \nabla_m A^\alpha,$$

i.e. $\quad \alpha \alpha_m A^i_j + \alpha \nabla_m A^i_j = \alpha \lambda_m A^i_j + A^i_j (\alpha \alpha_m - A^\alpha \nabla_m \lambda_\alpha),$

That is to say, we have

$$\alpha \nabla_m A^i_j = (\alpha \lambda_m - A^\alpha \nabla_m \lambda_\alpha) A^i_j,$$

or, being $a \neq 0$, we can see

$$\nabla_m A^i_j = \delta_m A^i_j, \qquad \delta_m \overset{def}{=\!=} \lambda_m - A^\alpha \nabla_m \lambda_\alpha / \alpha.$$

This completes the proof.

REFERENCES

[1] K. Takano, Recurrent affine motion in $S\text{–}AK^*_N$ spaces, *Tensor, N.S.*, 17 (1966), 321-326.

[2] K. Takano, On the existence of affine motion in a space with recurrent curvature, *Tensor, N.S.*, 17 (1966), 68-73.

Jñānābha, Vol. 30, 2000

ON SOME MULTIDIMENSIONAL INTEGRAL TRANSFORMS OF SRIVASTAVA AND PANDA'S *H*-FUNCTION OF SEVERAL COMPLEX VARIABLES

By

R.C. Singh Chandel and Kamlendra Kumar

Department of Mathematics, D.V. Postgraduate College,

Orai-285001, U.P., India

(Received : July 10, 2000)

ABSTRACT

In the present paper, we obtain multidimensional Laplace transforms and Whittaker transforms of Srivastava and Panda's *H*-function of several complex variables ([9], [10] ; also see [11], p. 251).

1. Introduction. Chandel [1] introduced multidimensional Laplacian operator to give integral representations of multiple hypergeometric functions of several variables $F^{(n)}_A$, $F^{(n)}_B$, $F^{(n)}_D$, of Lauricalla [7]. Chandel [2], further used this operator to give integral representations of multiple hypergeometric functions ${}^{(k)}_{(1)}E^{(n)}_D$, ${}^{(k)}_{(2)}E^{(n)}_D$. of Exton ([5],[6]). Also Chandel and Dwivedi ([3],[4]) introduced multidimensional Whittaker transforms of Lauricella multiple hypergeometric functions of sevreral variables [7], Exton ([5],[6]) and most generalized multiple hypergeometric function of Srivastava and Daoust [8] (also see Srivastava and Manocha [12, p. 64, [18], [19], [20]).

In the present paper, we further extend the above works to obtain the multidimensional Laplace transforms and Whittaker transforms of the most generalized Srivastava and Panda's *H*-function of several complex variables ([9], [10] ; also see [11, p.251]).

2. Laplacian Operator. Chandel [1] introduced multidimensional Laplacian operator

$$(2.1)\quad L^{(\lambda,\mu)}_{\alpha_1,\dots,\alpha_n}\{\ \} = \frac{\Gamma(\alpha_1+\dots+\alpha_n)\,\lambda^{\alpha_1+\dots+\alpha_n+\mu}}{\Gamma(\alpha_1)\dots\Gamma(\alpha_n)\ \Gamma(\alpha_1+\dots+\alpha_n+\mu)}\int_0^\infty\!\!\dots\int_0^\infty e^{-\lambda(x_1+\dots+x_n)}(x_1+\dots+x_n)^\mu$$

$$x_1^{\alpha_1-1}\dots x_n^{\alpha_n-1}\{\ \}\,dx_1\dots dx_n,$$

where $Re\,(\alpha_j)>0, j=1,\dots,n$, $Re\,(\lambda)>0$ and $Re\,(\alpha_1+\cdots+\alpha_n+\mu)>0$.
Here we give following additional applications of the above operator

$$(2.2)\quad L^{(\lambda,\mu)}_{\alpha_1,\dots,\alpha_n}\Big\{H^{0,\gamma:(\mu',\upsilon');\dots;(\mu^{(n)},\upsilon^{(n)})}_{A,C:(B',D');\dots;(B^{(n)},D^{(n)})}\Big(\begin{matrix}[(a):(\theta',\dots,\theta^{(n)})]:[(b'):\phi'];\dots;[(b^{(n)}):\phi^{(n)}];\\ [(c):(\psi',\dots,\psi^{(n)})]:[(d'):\delta'];\dots;[(d^{(n)}):\delta^{(n)}];\end{matrix}$$

$$u_1x_1^{\sigma_1}(x_1+\dots+x_n)^{\upsilon_1},\dots,u_n x_n^{\sigma_n}(x_1+\dots+x_n)^{\upsilon_n}\Big)\Big\}$$

$$= \frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)\ \Gamma(\alpha_1+\ldots+\alpha_n+\mu)} H^{0,\gamma+1\,:\,(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+1,C+1\,:\,[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\psi',\ldots,\psi^{(n)}],\end{matrix}\right.$$

$$[1-(\alpha_1+\ldots+\alpha_n+\mu):\sigma_1+\upsilon_1,\ldots,\sigma_n+\upsilon_n]:[(b'):(\phi')],\ [1-\alpha_1:\sigma_1];\ldots;[(b^{(n)}):(\phi^{(n)})],\ [1-\alpha_n:\sigma_n];$$
$$[1-(\alpha_1+\ldots+\alpha_n)\ :\sigma_1,\ldots,\sigma_n]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];$$
$$\left. u_1/\lambda^{\sigma_1+\upsilon_1},\ldots,u_n/\lambda^{\sigma_n+\upsilon_n}\right),$$

where $Re\ (\lambda)>0$, $Re\ (\alpha_1+\ldots+\alpha_n+\mu)>0$, $Re\ (\alpha_i)>0$, σ_i, υ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

$$|arg\ u_i|<\pi\Delta_i/2,\ i=1,\ldots,n,$$

and H-is Srivastava and Panda's H-function of several complex variables ([9],[10] ; also see [11], p. 251).

3. Generalized Whittaker Tarnsforms. Chandel and Dwivedi ([3],[4]) introduced and stutied generalized Whittaker transform

$$(3.1)\ \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\ \{\}$$
$$= \frac{\lambda^{\sigma+\alpha_1+\ldots+\alpha_n}(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\Gamma(\sum_{i=1}^{n}(\alpha_i)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\,(\prod_{i=1}^{n}\Gamma(\alpha_i)}\int_0^\infty\ldots\int_0^\infty \eta_1^{\alpha_1-1}\ldots\eta_n^{\alpha_n-1}(\eta_1+\ldots+\eta_n)^{\sigma}$$
$$e^{-\lambda(\eta_1+\ldots+\eta_n)/2}\,W_{\mu,\upsilon}\,[\lambda(\eta_1+\ldots+\eta_n)]\{\}\,d\eta_1\ldots d\eta_n,$$

where $Re\ (\lambda)>0$, $Re\ (\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)>0$ *and* $Re\ (\alpha_i)>0$, $i=1,\ldots,n$. As an application of obove transform, we derive

$$(3.2)\ \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\left\{H^{0,\gamma\,:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$
$$\left.\left. u_1\,x_1^{\sigma_1}(x_1+\ldots+x_n)^{\rho_1},\ldots,u_n\,x_n^{\sigma_n}(x_1+\ldots+x_n)^{\rho_n}\right)\right\}$$
$$= \frac{\Gamma(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\,\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1+\ldots+\alpha_n)}\ H^{0,\gamma+2\,:\,(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2\,:\,[B'+1,D'];\ldots;[B^{(n)}+1,\ D^{(n)}]}$$
$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)})],\ :[1/2-(\sigma\pm\upsilon)\ :\sigma_1+\rho_1,\ldots,\sigma_n+\rho_n];\\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\ldots+\alpha_n)\ :\sigma_1+\rho_1,\ldots,\sigma_n+\rho_n],[1-(\alpha_1+\ldots+\alpha_n):\sigma_1,\ldots,\sigma_n];\end{matrix}\right.$$
$$[(b'):(\phi')],\ [1-\alpha_1:\sigma_1];\ldots;[(b^{(n)}):(\phi^{(n)})],\ [1-\alpha_n:\sigma_n];$$
$$[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\qquad \left.u_1/\lambda^{\sigma_1+\upsilon_1},\ldots,u_n/\lambda^{\sigma_n+\upsilon_n}\right)$$

provided $Re\ (\lambda)>0$, $Re\ (\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)>0$, $Re\ (\alpha_i)>0$, σ_i, ρ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

and

$$|arg\, u_i| < \pi\,\Delta_i/2\,,\ i=1,\ldots,n.$$

Special cases of (3.2)

Case (a) *For* $\sigma_1=\ldots=\sigma_n=0$, we derive from (3.2)

$$(3.3)\ \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\left\{H^{0,\gamma\,:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$
$$\left.\left. u_1(x_1+\ldots+x_n)^{\rho_1},\ldots,u_n(x_1+\ldots+x_n)^{\rho_n}\right)\right\}$$
$$=\frac{\Gamma(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\,\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}\ H^{0,\gamma+2\,:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A+2,C+1\,:\,[B',D'];\ldots;[B^{(n)},D^{(n)}]}$$
$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[1/2-(\sigma\pm\upsilon):\rho_1,\ldots,\rho_n]:[(b'):(\phi')];\ldots;[(b^{(n)}):(\phi^{(n)})];\\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\ldots+\alpha_n):\rho_1,\ldots,\rho_n]:[(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\right.$$
$$\left. u_1/\lambda^{\rho_1},\ldots,u_n/\lambda^{\rho_n}\right).$$

where $Re\,(\lambda)>0$, $Re\,(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)>0$, $Re\,(\alpha_i)>0$, σ_i,ρ_i are positive numbers

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

and $|arg\,u_i|<\pi\,\Delta_i/2\,;\ i=1,\ldots,n.$

Case (b) For $\rho_1=\ldots=\rho_n=0$, (3.2) further gives

$$(3.4)\ \Omega^{\lambda,\mu,\upsilon}_{\alpha_1,\ldots,\alpha_n;\sigma}\left\{H^{0,\gamma\,:\,(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C\,:\,[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta';\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}):\phi^{(n)}];\\ [(c):\psi';\ldots;\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\right.\right.$$
$$\left.\left. u_1x_1^{\sigma_1},\ldots,u_nx_n^{\sigma_n}\right)\right\}$$
$$=\frac{\Gamma(\sigma+1-\mu+\alpha_1+\ldots+\alpha_n)\,\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\sigma+1/2+\alpha_1+\ldots+\alpha_n\pm\upsilon)\,\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}\ H^{0,\gamma+2\,:\,(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2\,:\,[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]}$$
$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[1/2-(\sigma\pm\upsilon):\sigma_1,\ldots,\sigma_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],[\mu-(\sigma+\alpha_1+\ldots+\alpha_n):\sigma_1,\ldots,\sigma_n];[1-(\alpha_1+\ldots+\alpha_n):\sigma_1,\ldots,\sigma_n]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\phi'],[1-\alpha_1:\sigma_1];\ldots;[(b^{(n)}):\phi^{(n)}],[1-\alpha_n:\sigma_n];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\quad u_1/\lambda^{\sigma_1},\ldots,u_n/\lambda^{\sigma_n}\right).$$

4. Other multidimensional Whittaker transform $T^{\lambda,\mu,\upsilon}_{\beta_1,\ldots,\beta_n;\sigma}$

Chandel and Dwivedi [4] introduced and studied the multidimensional Whittaker transform

$$T^{\lambda,\mu,\upsilon}_{\beta_1,\ldots,\beta_n;\sigma}\ \{\ \},$$
$$=\frac{K\,\lambda^{\sigma+\beta_1+\ldots+\beta_n}\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)\Gamma(\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)\prod_{j=1}^{\upsilon}\Gamma\beta_i}$$
$$\int_0^\infty\ldots\int_0^\infty e^{-\lambda/2\left[\sum_{j=1}^{n}(\alpha_1{}^jx_1+\ldots+\alpha_n{}^jx_n)\right]}\prod_{j=1}^{n}(\alpha_1{}^jx_1+\ldots+\alpha_n^jx_n)^{\beta_j-1}\left[\sum_{j=1}^{n}(\alpha_1{}^jx_1+\ldots+\alpha_n{}^jx_n)\right]^{\sigma}$$

$W_{\mu,\upsilon}[\lambda\sum_{j=1}^{n}(\alpha_1^j x_1+\ldots+\alpha_n^j x_n)]\{\}\,dx_1\ldots dx_n.$

where $Re(\lambda)>0$, $Re(\sigma+1/2+\beta_1+\ldots+\beta_n\pm\upsilon)>0$, $Re(\sigma+1-\mu+\beta_1+\ldots+\beta_n)>0$, $Re(\beta_j)>0$ $j=1,\ldots,n$ and

$$K=\begin{vmatrix}\alpha_1^1\,\alpha_2^1 & \ldots & \alpha_n^1\\ \alpha_1^2\,\alpha_2^2 & \ldots & \alpha_n^2\\ \ldots & \ldots & \ldots\\ \alpha_1^n\,\alpha_2^n & \ldots & \alpha_n^n\end{vmatrix}\neq 0.$$

As additional applications of above transform, we derive

(4.2) $$T^{\lambda,\mu,\upsilon}_{\beta_1,\ldots,\beta_n;\sigma}\left\{H^{0,\gamma:(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a),:\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c),:\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}:\delta^{(n)}];\end{matrix}\right.\right.$$
$$\left.\left. u_1(\alpha_1^1x_1+\ldots+\alpha_n^1x_n)^{\eta_1},\ldots,u_n(\alpha_1^nx_1+\ldots+\alpha_n^nx_n)^{\eta_n}\right)\right\}$$
$$=\frac{\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)\,\Gamma(\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)\,\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\;H^{0,\gamma+2:(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1);}_{A+2,C+2:[B'+1,D'];\ldots;[B^{(n)}+1,\,D^{(n)}];}$$
$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)})],\,[1/2-(\sigma\pm\upsilon+\beta_1+\ldots+\beta_n):\eta_1,\ldots,\eta_n]:\\ [(c):\psi',\ldots,\psi^{(n)}],\,[\mu-(\sigma+\beta_1+\ldots+\beta_n):\eta_1,\ldots,\eta_n],\,[1-(\beta_1+\ldots+\beta_n):\eta_1,\ldots,\eta_n]:\end{matrix}\right.$$
$$\left.\begin{matrix}[(b'):\phi'],\,[1-\beta_1:\eta_1];\ldots;[b^{(n)}:\phi^{(n)}],[1-\beta_n:\eta_n];\\ [(d'):\delta'];\ldots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\quad u_1/\lambda^{\eta_1},\ldots,u_n/\lambda^{\eta_n}\right),$$

where $Re(\lambda)>0$, $Re(\sigma+1/2+\beta_1+\ldots+\beta_n\pm\upsilon)>0$, $Re(\sigma+1-\mu+\beta_1+\ldots+\beta_n)>0$,

$$K=\begin{vmatrix}\alpha_1^1\,\alpha_2^1 & \ldots & \alpha_n^1\\ \alpha_1^2\,\alpha_2^2 & \ldots & \alpha_n^2\\ \ldots & \ldots & \ldots\\ \alpha_1^n\,\alpha_2^n & \ldots & \alpha_n^n\end{vmatrix}\neq 0,$$

$Re(\beta_i)>0$, η_i are positive numbers,

$$\Delta_i=-\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

and $|arg\,u_i|<\pi\Delta_i/2$, $i=1,\ldots,n.$

(4.3) $$T^{\lambda,\mu,\upsilon}_{\beta_1,\ldots,\beta_n;\sigma}\left\{H^{0,\gamma:(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}:\delta^{(n)}];\end{matrix}\right.\right.$$
$$\left.\left. u_1[\sum_{j=1}^{n}(\alpha_1^jx_1+\ldots+\alpha_n^jx_n]^{\zeta_1},\ldots,u_n[\sum_{j=1}^{n}(\alpha_1^jx_1+\ldots+\alpha_n^jx_n]^{\zeta_n}\right)\right\}$$
$$=\frac{\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)}\,H^{0,\gamma+2:(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A+2,C+1:[B',D'];\ldots;[B^{(n)},\,D^{(n)}]}$$
$$\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)})],\,[1/2-(\sigma\pm\upsilon+\beta_1+\ldots+\beta_n):\zeta_1,\ldots,\zeta_n]:[(b'):(\phi')];\ldots;[b^{(n)}):(\phi^{(n)})];\\ [(c):\psi',\ldots,\psi^{(n)}],\,[\mu-(\sigma+\beta_1+\ldots+\beta_n):\zeta_1,\ldots,\zeta_n]:[(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\right.$$

$$u_1/\lambda^{\zeta_1},\ldots,u_n/\lambda^{\zeta_n}\Big) ,$$

where all conditions of (4.2) are satisfied and ζ_i are positive numbers, $i=1,\ldots,n$.

The results (4.2) and (4.3) can be further genealized as

$$(4.4)\; T^{\lambda,\mu,\upsilon}_{\beta_1,\ldots,\beta_n;\sigma}\Big\{H^{0,\gamma:(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\phi'];\ldots;[(b^{(n)}:\phi^{(n)}];\\ [(c):\psi',\ldots,\psi^{(n)}]:[(d'):\delta'];\ldots;[(d^{(n)}:\delta^{(n)}];\end{matrix}$$

$$u_1(\alpha_1{}^1x_1+\ldots+\alpha_n{}^1x_n)^{\eta_1}[\textstyle\sum_{j=1}^{n}(\alpha_1{}^jx_1+\ldots+\alpha_n{}^jx_n)]^{\zeta_1},\ldots,u_n(\alpha_1{}^nx_1+\ldots+\alpha_n{}^nx_n)^{\eta_n}[\textstyle\sum_{j=1}^{n}(\alpha_1{}^jx_1+\ldots+\alpha_n{}^jx_n)]^{\zeta_n}\Big)\Big\}$$

$$= \frac{\Gamma(\sigma+1-\mu+\beta_1+\ldots+\beta_n)\;\Gamma(\beta_1+\ldots+\beta_n)}{\Gamma(\sigma+1/2\pm\upsilon+\beta_1+\ldots+\beta_n)\,\Gamma(\beta_1)\ldots\Gamma(\beta_n)}\,H^{0,\gamma+2:(\mu',\upsilon'+1);\ldots;(\mu^{(n)},\upsilon^{(n)}+1)}_{A+2,C+2:[B'+1,D'];\ldots;[B^{(n)}+1,D^{(n)}]}$$

$$\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}),\,[1/2-(\sigma\pm\upsilon+\beta_1+\ldots+\beta_n):\eta_1+\zeta_1,\ldots,\eta_n+\zeta_n]:[(b'):(\phi')],[1-\beta_1,\eta_1]\\ [(c):\psi',\ldots,\psi^{(n)}],\,[\mu-(\beta_1+\ldots+\beta_n+\sigma):\eta_1+\zeta_1,\ldots,\eta_n+\zeta_n],\,[1-(\beta_1+\ldots+\beta_n):\eta_1,\ldots,\eta_n]\end{matrix}$$

$$\begin{matrix};\ldots;[b^{(n)}):(\phi^{(n)})],[1-\beta_n,\eta_n];\\ [(d'):(\delta')];\ldots;[(d^{(n)}):(\delta^{(n)})];\end{matrix}\quad u_1/\lambda^{\eta_1+\zeta_1},\ldots,u_n/\lambda^{\eta_n+\zeta_n}\Big),$$

where η_i and ζ_i, $i=1,\ldots,n$ are all positive numbers, and all other conditions of (4.2) are satisfied.

5. Another Multidimensional Whittaker Transform : Chandel and Dwivedi [4] introduced and studied another multidimensional Whittaker transform

$$(5.1)\; H^{\mu_1,\ldots,\mu_n;\upsilon_1,\ldots,\upsilon_n}_{\beta_1,\ldots,\beta_n}\{\}=K\prod_{i=1}^{n}\frac{\Gamma(1+\beta_i-\mu_i)}{\Gamma(1/2+\beta_i\pm\upsilon_i)}\int_0^\infty\ldots\int_0^\infty exp[-1/2\sum_{i=1}^{n}(\alpha_1{}^ix_1+\ldots+\alpha_n{}^ix_n)]$$

$$\prod_{i=1}^{n}(\alpha_1{}^ix_1+\ldots+\alpha_n{}^ix_n)^{\beta_i-1}\,W_{\mu_i,\upsilon_i}(\alpha_1{}^ix_1+\ldots+\alpha_n{}^ix_n)\{\;\}\;dx_1\ldots dx_n,$$

where $Re\,(1+\beta_i-\mu_i)>0$, $Re\;(1/2+\beta_i\pm\upsilon_i)>0$, $i=1,\ldots,n$ and

$$K=\begin{vmatrix}\alpha_1{}^1\,\alpha_2{}^1 & \ldots & \alpha_n{}^1\\ \alpha_1{}^2\,\alpha_2{}^2 & \ldots & \alpha_n{}^2\\ \ldots\;\ldots & \ldots & \ldots\\ \alpha_1{}^n\,\alpha_2{}^n & \ldots & \alpha_n{}^n\end{vmatrix}\neq 0,$$

As an its additional application, we derive

$$(5.2)\; H^{\mu_1,\ldots,\mu_n;\upsilon_1,\ldots,\upsilon_n}_{\beta_1,\ldots,\beta_n}\Big\{H^{0,\gamma:(\mu',\upsilon');\ldots;(\mu^{(n)},\upsilon^{(n)})}_{A,C:[B',D'];\ldots;[B^{(n)},D^{(n)}]}\Big(\begin{matrix}[(a),:\theta',\ldots,\theta^{(n)}]:\\ [(c),:\psi',\ldots,\psi^{(n)}]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'];\ldots;[b^{(n)}):\phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\;u_1(\alpha^1{}_1x_1+\ldots+\alpha^1{}_nx_n)^{\eta_1},\ldots,u_n(\alpha_1{}^nx_1+\ldots+\alpha_n{}^nx_n)^{\eta_n}\Big)\Big\}$$

$$=\prod_{i=1}^{n}\frac{\Gamma(1+\beta_i-\mu_i)}{\Gamma(1/2+\beta_i\pm\upsilon_i)}\Big\{\mathrm{H}^{0,\gamma:(\mu':\upsilon'+2);\ldots;(\mu^{(n};\upsilon^{(n)}+2)}_{A,C:(B'+2,D'+1);\ldots;(B^{(n)}+2,D^{(n)}+1)}\Big(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\psi',\ldots,\psi^{(n)}]:\end{matrix}$$

$$\begin{matrix}[(b'):\phi'],\,[1/2-(\beta_1\pm\upsilon_1):\eta_1];\ldots;[b^{(n)}):\phi^{(n)}],[1/2-(\beta_n\pm\upsilon_n):\eta_n];\\ [(d'):\delta'],\,[\mu_1-\beta_1:\eta_1];\ldots;[(d^{(n)}):(\delta^{(n)})];[(\mu_n-\beta_n:\eta_n];\end{matrix}\quad u_1,\ldots,u_n\Big),$$

where $Re\,(1+\beta_i-\mu_i)>0$, $Re\,(1/2+\beta_i\pm\upsilon_i)>0$, $\eta_i>0$,

$$K=\begin{vmatrix}\alpha_1^{\,1}\,\alpha_2^{\,1} & \dots & \alpha_n^{\,1}\\ \alpha_1^{\,2}\,\alpha_2^{\,2} & \dots & \alpha_n^{\,2}\\ \dots\;\dots & \dots & \dots\\ \alpha_1^{\,n}\,\alpha_2^{\,n} & \dots & \alpha_n^{\,n}\end{vmatrix}\neq 0,$$

$$\Delta_i = -\sum_{j=\gamma+1}^{A}\theta_j^{(i)}+\sum_{j=1}^{\upsilon^{(i)}}\phi_j^{(i)}-\sum_{j=\upsilon^{(i)}+1}^{B^{(i)}}\phi_j^{(i)}$$
$$-\sum_{j=1}^{C}\psi_j^{(i)}+\sum_{j=1}^{\mu^{(i)}}\delta_j^{(i)}-\sum_{j=\mu^{(i)}+1}^{D^{(i)}}\delta_j^{(i)}>0,$$

and

$|arg\,u_i|<\pi\Delta_i/2\ ,i=1,..,n$

REFERENCES

[1] R.C.S. Chandel, The products of certain classical polynomiols and the generlized Laplacian operator, *Ganita*, **20** (1969), 79-87 *Corigendum* **23** (1972), 90.

[2] R.C.S. Chandel, On some multiple hypergeometric functions related to Lauricella funcations, *Jñānābha, Sect. A* **3** (1973), 119-136.

[3] R.C.S. Chandel and B.N.Dwivedi, Generalized Whittaker transforms of hypergeometric funcations of several variables, *Bull. Inst. Math. Acad. Sinica*, **8** No. 4 (1980), 595-601.

[4] R.C.S. Chandel and B.N Dwivedi, multidimentional Whittaker transforms, *Indian J.Math.* 24 (1982), 49-53.

[5] H. Exton, On two multiple hypergeometric functions related to Lauricella's $F_D^{(n)}$, *Jñānābha, Sect. A,* **2** (1972), 59-73.

[6] H-Exton, *Multiple Hypergeometric Functions and Applications*, John Wiley and Sons, New York, 1976.

[7] G. Lauricella, Sulle funzioni ipergeometriche a più variabili, *Rend, Circ. Mat. Palermo*, **7** (1893), 111-158.

[8] H.M. Srivastava and M.C. Daoust, Certain generalized Neumann expansions associated with the Kampé de Fériet function, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** = *Indag. Math.* **31** (1969), 449-457.

[9] H.M. Srivastava and R. Panda, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265-274.

[10] H.M. Srivastava and R. Panda, Expansion theorems for the *H*-function of several complex variables, *J. Reine Angew. Math.*, **288** (1976), 129-145.

[11] H.M. Srivastava, K.C. Gupta and S.P. Goyal, *The H-Functions of One and Two Variables with Applications*, South Asian Publishers, New Delhi and Madras, 1982.

[12] H.M. Srivastava and H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester) John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

Jñānābha, Vol. 30, 2000

ON DISTRIBUTIONAL STRUVE TRANSFORMATION

By

Vijay Kumar

Department of Mathematics

P.T.P.S. College, Patratu, Hazaribagh - 829119, India

and

N.K. Agrawal*

University Professor and Head, Department of Mathematics,

Marwari College, Ranchi-834 001, India

(Received : October 4, 1999)

ABSTRACT

The classical Struve transformation has been extended to a class of generalized functions. The generalized Struve transformation, so defined is seen to be a smooth function.

1. Introduction. The integral transfrom

$$(H_v f)(x) = \int_0^\infty \sqrt{(xt)}\, H_v(xt) f(t)\, dt \qquad \text{... (1.1)}$$

where $H_v(x)$ is the Struve function, has been studied briefly in [5, 8.4]. Recently, Love, [2], Rooney, [4] and Heywood and Rooney [1] have studied it in details.

The aim of the present work is to extend the Struve transform (1.1) to a class of generalized functions. It will be shown that the generalized Struve transform is a smooth function.

R.S. Pathak and J.N. Pandey [3] have extended the Hardy transformation given by

$$f(x) = \int_0^\infty F_v(tx)\, tdt\, C_v(ty) y\, f(y) dy, \qquad \text{... (1.2)}$$

where $C_v(z) = \cos p\pi\, J_v(z) + \sin p\pi\, Y_v(z)$

$$\text{and} \quad F_v(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{v+2p+2m}}{\Gamma(p+m+1)\,\Gamma(p+m+v+1)}.$$

It has been shown there that the Struve transform (1.1) is a special case of (1.2) when $p = 1/2$. The present paper deals the transform (1.1) Indepenently and differently.

2. Definition and properties of Struve's function. In this section, we quote some results on the Struve's function which we need hereafter of and on.

The Struve's function $H_v(z)$, of oreder v. is defined by the equations

Mailing Address : *D-121, Harmu Housing Colony, Ranchi - 834 012, India*

[6, p. 328].

$$H_\upsilon(z) = \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^1 (1-t^2)^{\upsilon-1/2} \sin zt.\, dt \qquad \text{... (2.1)}$$

$$= \frac{(z/2)^\upsilon}{\Gamma(\upsilon+1/2)\,\Gamma(1/2)} \int_0^{\pi/2} \sin(z\cos\theta)\, \sin^{2\upsilon}\theta\, d\theta, \qquad \text{... (2.2)}$$

provided that $Re(\upsilon) > -1/2$, or equivalently

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\upsilon+2m+1}}{\Gamma(m+3/2)\,\Gamma(\upsilon+m+3/2)}. \qquad \text{... (2.3)}$$

Also, [6,p. 328]

$$\frac{d}{dz}\{z^\upsilon H_\upsilon(z)\} = z^\upsilon H_{\upsilon-1}(z), \qquad \text{... (2.4)}$$

$$\frac{d}{dz}\{z^{-\upsilon} H_\upsilon(z)\} = \frac{1}{2^\upsilon\,\Gamma(\upsilon+3/2)\,\Gamma(1/2)} - z^{-\upsilon} H_{\upsilon+1}(z). \qquad \text{... (2.5)}$$

From [6, p. 337], we have

$$H_\upsilon(z) = o(z^{-1/2}) \quad (\upsilon \le 1/2)$$
$$= o(z^{\upsilon-1}) \quad (\upsilon \ge 1/2) \text{ as } z \to \infty \qquad \text{.... (2.6)}$$

Also, $H_\upsilon(z) = o(z^{\upsilon+1})$ as $z \to 0$ .... (2.7)

3. The testing function space $H_a(I)$ and its dual $H'_a(I)$.

Let I be the interval $(0,\infty)$ and a, a fixed positive number. An infinitely differentiable function $\phi(x)$ defined over I is said to belong to $H_a(I)$ if

$$\gamma_k(\phi) = \sup_{0<t<\infty} |e^{-at}(tD)^k \phi(t)| < \infty \qquad \text{... (3.1)}$$

for $k = 0, 1, 2, \ldots$ and $D = d/dt$.

The topology on $H_a(I)$ is defined by means of the separating collection of semi-norms $\{\gamma_k\}_{k=0}^\infty$. A sequence $\{\phi_\upsilon\}_{\upsilon=1}^\infty$ is said to converge in $H(I)$ to the limit ϕ if $\gamma_k(\phi_\upsilon - \phi) \to 0$ as $\upsilon \to \infty$ for each $k = 0, 1, 2, \ldots$.
A sequence $\{\phi_\upsilon\}_{\upsilon=1}^\infty$ is said to be a Cauchy sequence in $H(I)$ if $\gamma_k(\phi_\upsilon - \phi_\mu)$ goes to zero as υ, μ both go to infinity independently of each other. It can be readily seen that $H_a(I)$ is locally convex, sequentially complete, Hausdorff topological vector space. The dual of $H_a(I)$ will be represented by $H'_a(I)$.

Lemma 3.1 . For $a > 0$, $\upsilon > -1/2$ and $t, x > 0$, then for a fixed $x > 0$, $h(xt) = \sqrt{(xt)}\, H_\upsilon(xt) \in H_\upsilon(I)$, $H_\upsilon(z)$ being Struve fucntion as defined in Section 2.

Proof. From differential properties of Struve functions, we have

$$|e^{-at}(tD_t)^k h(xt)|$$

$$= |e^{-at} \sum_{j=0}^{k} a_j(\upsilon)\,(xt)^{(1/2)+j} H_{\upsilon-j}(xt)| < \infty,$$

where $a_j(\upsilon)$ is a polynomial in υ.

Lemma 3.2. For $a > 0$, $\upsilon = -1/2$ and $t, x > 0$,

$$\frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\,H_\upsilon(xt)] \in H_a(I).$$

Proof. We have

$$|e^{-at}(tD_t)^k \frac{\partial^m}{\partial x^m}[\sqrt{}^n(xt)^k H_\upsilon(xt)]|$$

$$= |e^{-at}\Big(\sum_{j=0}\sum_{p=0} A_{j.p}(\upsilon).\, x^{-m}(xt)^{j+p+1/2} H_{\upsilon-j-p}(xt)\Big)| < \infty,$$

for a fixed $x > 0$, $\upsilon > -1/2$.

We now enlist some properties of the spaces $H_a(I)$ and its dual $H'_a(I)$.

Property 3.1. For $0 < a < b < 1/2$, $D(I) \subset H_b(I) \subset H_a(I) \subset E(I)$, all inclusions being continuous. Moreover as $D(I)$ is dense in $E(I)$, $H_a(I)$ is dense in $E(I)$.

Property 3.2. The dual $H'_a(I)$ equipped with the usual weak topology is Hausdorff locally convex, sequentially complete space of generalized functions.

For $f \in H'_a(I)$ there exists $C > 0$ and an integer r (C and r depending on f) such that

$$|<f, \phi>| \leq C \max_{0 \leq k \leq r} \gamma_k(\phi).$$

Property 3.3: A locally integrable function f in I such that $e^{at}f(t)$ is absolutely integrable on I, gives rise to a regular generalized function of $H'_a(I)$ through

$$|<f, \phi>| = \int_0^\infty f(t)\,\phi(t)\,dt, \ \forall\, \phi \in H_a(I).$$

Property 3.4. The differential operator (tD_t) is a continuous linear mapping from $H_a(I)$ into itself. $(t\,D_t)'$, the adjoint of $(t\,D_t)$ maps continuously $H'_a(I)$ into itself.

4. The generalized transform. For $f \in H'_a(I)$, the generalized Struve transform is defined by

$$s[f] = F(x) = <f(t), \sqrt{(xt)}\,H_\upsilon(xt)> \qquad \text{... (4.1)}$$

where x is a non-zero real number and $t > 0$. From Lemma 3.1 we know that for fixed $x > 0$,

$$\sqrt{(xt)}\,H_\upsilon(xt) \in H_a(I),$$

where $\upsilon > -1/2$, $a > 0$. The relation (4.1) is meaningful.

Theorem 4.1. For real $x \neq 0$, let $F(x)$ be defined by (4.1). Then $F(x)$ is infinitely differentiable and that

$$F^m(x) = <f(t), \frac{\partial^m}{\partial x^m}[\sqrt{(xt)}\,H_a(xt)] \qquad \text{... (4.2)}$$

for all real $x \neq 0$ and $m=1,2, \ldots$.

Proof. By Lemma 3.2, it follows that $\dfrac{\partial^m}{\partial x^m}[\sqrt{(xt)}\, H_a(xt)] \in H_a(I)$. Hence, we need merely to prove (4.2), what we do through an inductive argument. We assume that (4.2) is true for m replaced by $(m-1)$. It is true by definition for $m = 0$. Keeping x fixed and $\partial x \neq 0$, consider

$$\frac{1}{\partial x}[D_x^{m-1} F(x+\partial x) - D_x^{m-1} F(x) - < f(t), D_x^m \sqrt{(xt)}\, H_\upsilon(xt) >$$

$$= \frac{1}{\partial x}[D_x^{m-1} < f(t), \sqrt{[(x+\partial x)t]}\, H_\upsilon(\overline{x+\partial x}\, t)> - D_x^{m-1} < f(t), \sqrt{(xt)} H_\upsilon(xt) >]$$
$$- < f(t), D_x^m [\sqrt{(xt)}\, H_\upsilon(xt)] >$$

$$= < f(t), \frac{D_x^{m-1} h\, (\overline{x+\partial x\, t})\, D_x^{m-1} h\,(xt)}{\partial x} - D_x^m h\,(xt) >$$

(writing $h(xt)$ for $\sqrt{(xt)}\, H_\upsilon(xt)$.)

$$= < f(t), A_{\partial x}(t) > (say) \qquad \ldots (4.3)$$

where,

$$A_{\partial x}(t) = \frac{D_x^{m-1} h\, (\overline{x+\partial x\, t})\, D_x^{m-1} h\,(xt)}{\partial x} - D_x^m h\,(xt)$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} + D_u^m h\,(ut)\, du - \frac{1}{\partial x}\int_x^{x+\partial x} D_x^m\, h\,(xt)\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} [D_u^m h\,(ut) - D_x^m\, h\,(xt)]\, du$$

$$= \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1}\, h\,(\eta t)\, d\eta.$$

Now

$$| e^{-at}\, (tD_t)^k A_{\partial x}(t) |$$

$$= | e^{-at}\, \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1}\ \{(tD_t)^k h(\eta t)\}\, d\eta\, |$$

$$= | e^{-at}\, \frac{1}{\partial x}\int_x^{x+\partial x} du \int_x^u D_\eta^{m+1} \{ \Sigma\, a_j\,(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\, d\eta\, |$$

Let Δ denotes the interval $x - |\partial x| < n < x + |\partial x|$.

Then,

$$| e^{-at}\, (tD_t)^k A_{\partial x}(t) |$$

$$\leq \frac{|\partial x|}{2} e^{-at} \sup\, | D_\eta^{m+1} \sum_{j=0}^{k} a_j\,(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\ | \qquad \ldots (4.4)$$

Now, by Lemma 2.1,

$$e^{-at\ n\in\Delta}\ |D_\eta^{m+1} \sum_{j=0}^{k} a_j(\upsilon)(\eta t)^{1/2+j} H_{\upsilon-j}(\eta t)\ |$$

is bounded on $0 < t < \infty$, (taking $|\partial(x)| < 1$). Therefore, It follows from (4.4) that $A_{\partial x}(t)$ converges in $H(I)$ to zero as $\partial x \to 0$. Since $f \in H'(I)$, (4.3) converges to zero as $\partial x \to 0$. This completes our inductive proof of (4.2).

REFERENCES

[1] P. Heywood and P.G. Rooney, On the Struve Transformation, *SAIM J. Math. Anal.*, **25** (2) (1994) 450-461.

[2] E.R. Love, Inversion of the Struve Transform, *Fractional Calculus* (Glasgow, 1984), 75-86, *Res. Notes in Math.*, **138**, London, 1985.

[3] R.S. Pathak and J.N. Pandey, A distributional Hardy Transformation. *Proc. Camb. Phil. Soc.*, **76** (1974), 247-265.

[4] P.G. Rooney, On the Y_υ and H_υ transformations, *Canad. J. Math.*, **32** (5) (1980), 1021-1044.

[5] E.C. Titchmarsh, *Theory of Fourier Integrals*, Oxford, 1948.

[6] *G.N. Watson, Theory of Bessel Functions, Cambridge, 1962.*

Statement of ownership and particulars about the journal

JÑĀNĀBHA

1.	**Place of Publication**	D.V. Postgraduate College, Orai-285 001, U.P., India
2.	**Periodicity of publication**	Annual
3.	**Printer's Name**	Mr. Akhilesh Gupta
	Nationality	Indian
	Address	Computer House Orai-285001, U.P., India Ph. 05162-253654 -257717
4.	**Publisher's Name**	*Dr. R.C. Singh Chandel* For **Vijñāna Parishad of India**
	Nationality	Inidan
	Address	D.V. Postgraduate College Orai-285001, U.P., India
5.	**Editor's Name**	Dr. R.C. Singh Chandel
	Nationality	Inidan
	Address	D.V. Postgraduate College Orai-285001, U.P., India
6.	**Name and address of the individuals who own the journal and partners of share holders holding more than one percent of the total capital** Address:	Vijñāna Parishad of India D.V. Postgraduate College Orai-285001, U.P., India

I, *Dr. R.C. Singh Chandel* hereby declare that the particulars given above are true to the best of my knowledge and belief.

Sd./Dr. R.C. Singh Chandel
Publisher